Angular Momentum in General Relativity

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Abstract

This dissertation covers three topics in general relativity, each linked to notions of angular momentum. I first present a review of conservation laws in general relativity, discussing those associated with point-particle motion, field theories on a fixed background, and asymptotic symmetries for theories with no fixed background. This is followed by a review of the physics of spinning black holes, which are described by the Kerr spacetime. I lastly provide a review of gravitational waves, focusing on two physical phenomena that are relevant to this dissertation: extreme mass-ratio inspirals, an important source for future space-based gravitational wave detectors, and the gravitational wave memory effect, a permanent change in the separation of two freely-falling bodies caused by a burst of gravitational waves.

The first of the three topics covered in this dissertation concerns the Carter constant, a generalization of the notion of total angular momentum for point particles in the Kerr spacetime. The Carter constant is of great interest, as understanding how it evolves for an inspiralling particle is important for determining the gravitational wave signal. I first consider generalizations of the Carter constant to general field theories in the Kerr spacetime, and show that no generalizations of the Carter constant that depend only on the stress-energy tensor of the theory can be conserved. However, this result does not eliminate the possibility that such a generalization, not constructed from the stress-energy tensor, can exist in particular field theories. I next discuss how one can construct conserved currents in linearized gravity on a Kerr background which generalize the Carter constant. These currents generalize the Carter constant in the following sense: in the geometric optics limit, they are related to the Carter constants of individual gravitons.

For the second topic, I discuss generalizations of the gravitational wave memory effect. These generalizations, called "persistent gravitational wave observables", measure enduring effects following a burst of gravitational waves. This dissertation contains three examples of such persistent observables, as well as general techniques to calculate them, both in general

spacetimes and in exact plane gravitational wave spacetimes. The first example of a persistent observable is a generalization of geodesic deviation that allows for arbitrary acceleration. The second example is a holonomy around a closed loop in spacetime of a connection related to linear and angular momentum. Finally, the third example is an explicit procedure by which an observer could measure persistent effects using a spinning test particle.

The final topic considered is prescriptions for defining asymptotic charges in theories with no fixed background, and in particular angular momentum in Einstein-Maxwell theory. This is motivated by a strange result in electromagnetism, that the flux of angular momentum through null infinity, computed using the stress-energy tensor, depends on both radiative and Coulombic degrees of freedom. I first show that this situation carries over to electromagnetism on non-dynamical, asymptotically flat spacetimes for fluxes associated with the Lorentz symmetries in the asymptotic Bondi-Metzner-Sachs algebra. I then consider asymptotic charges (such as mass and angular momentum) in Einstein-Maxwell theory, where the metric is now dynamical. One could define these charges by using the same expressions as in vacuum general relativity, but such a prescription results in fluxes for the Lorentz charges that depend on Coulombic degrees of freedom, much as in the non-dynamical case. The correct approach is to use the prescription of Wald and Zoupas to compute the charges associated with any asymptotic symmetry on cross-sections of null infinity. The flux of this "new" notion of angular momentum depends only on radiative degrees of freedom and not Coulombic ones.

Biographical sketch

Alexander Grant was born on July 30th, 1991 at 12:36 AM in Ann Arbor, Michigan, to Barbara and Marshal Grant. He spent the next year and a half of his life in Michigan, after which he moved to Ithaca, New York, where he barely started preschool before moving again to Belmont, Massachusetts. In 2005, he attended Commonwealth High School, a private school in Boston where his mother taught. After briefly considering computer science and chemistry, a course on special relativity, taught by Dr. Farhad Riahi, convinced him to pursue the study of physics in college.

In 2009, he attended University of Chicago, where his first exposure to oscilloscopes convinced him to pursue theory. In 2012, he participated in an REU program in the Department of Mathematics, completing a project on Preissman's theorem in Riemannian geometry. At the time of writing, his write-up for this project is cited in the Wikipedia article on this theorem. In 2013, he graduated with a Bachelor of Arts in Physics, with honors—not due to any research achievements, but merely for taking a sufficient number of graduate courses. One of these courses, a class on general relativity taught by Prof. Robert Wald, rekindled his passion for relativistic physics from his high school days.

In 2013, he took a year off to pay off his student loans by working as a software engineer at Basis Technology in Cambridge, Massachusetts. In the fall of 2014, he joined the incoming class of graduate students in the Physics Department of Cornell University. Early in that semester, he began attending the group meetings of his future advisor, Prof. Éanna Flanagan, and has continued to do so. In addition to being a teaching assistant for a variety of courses, mostly the introductory classes for physics majors, he has spent much of his graduate career running board game nights for his fellow physics students and constructing large, wooden unicorns.

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Contents

Chapter 1

Introduction

General relativity is a theory of dynamical spacetimes, where "[s]pace tells matter how to move[; m]atter tells space how to move" [120]. In particular, these spacetimes are (in general) curved. Much of classical mechanics is based on the existence of a flat background on which bodies move—the promotion of this background to one that is curved, and even worse *dynamical*, deeply complicates, and in some cases completely negates, any intuition from flat spacetime.

This thesis considers a variety of topics in general relativity, each related (in some way) to how one generalizes the notion of angular momentum to a curved spacetime. Angular momentum, in introductory mechanics, is characterized by the following expression:[1]

$$\vec{L} = \vec{r} \times \vec{p}, \tag{1.1}$$

where $\vec{r}$ is the position of the particle relative to some origin and $\vec{p}$ its momentum. On the face of it, it is not clear how this would generalize to general relativity, as what is $\vec{r}$? While the position vector $\vec{r}$ in flat spacetime points along the unique line connecting the origin to the location of the particle, there is no unique curve which connects two points in a curved spacetime.

There are even worse problems when one considers the angular momentum of an extended body, where (in mechanics) the total angular momentum is given by an *integral* over the body:

$$\vec{L} = \int_V \vec{r} \times \vec{\wp} \, \mathrm{d}V, \tag{1.2}$$

[1] In this thesis, I will denote three-vectors (which rarely arise) with arrows, and four-vectors either with indices or in bold.

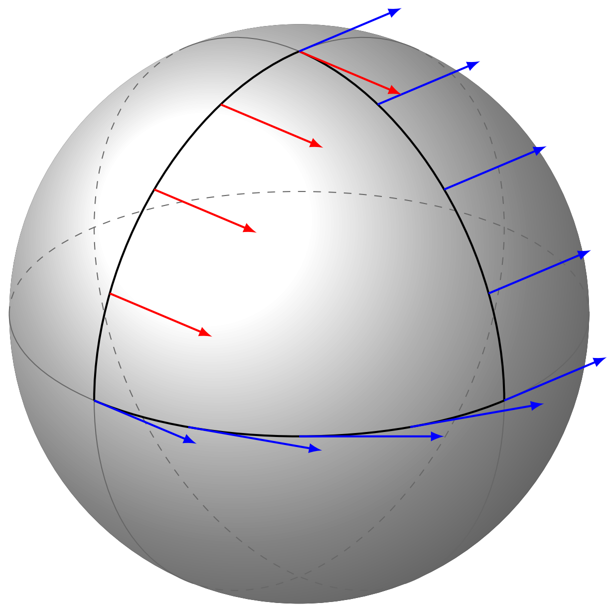

Figure 1.1: The process of moving a vector from one point to another on the surface of a sphere, which this figure demonstrates to be *path-dependent*. Starting at a common point at the left of this diagram, and moving up to north pole directly, yields a different vector (red) than first moving along the equator and then to the north pole (blue).

where $\vec{\wp}$ is the momentum density. The issue that arises here is that, in a curved spacetime, there is no unique way to add vectors at different points, as they live in different vector spaces. In a flat spacetime, one can move vectors throughout spacetime, keeping them parallel to themselves, in order to move them from one point to another, but in a curved spacetime such a procedure is *path-dependent* (for an example on a sphere, see figure 1.1). As such, in a curved spacetime it is not clear what is meant by the integral in equation (1.2).

The question of how one defines angular momentum in general relativity has several interesting applications. In classical mechanics, angular momentum (and its conservation) allows us to easily solve many problems that are otherwise impractical to consider, such as collisions and the Kepler problem of planetary motion. Moreover, the introduction of rigid bodies with intrinsic angular momentum introduces additional features into the dynamics of mechanical systems, such as the unintuitive behavior of gyroscopes. Generalizations of angular momentum to general relativity have a similar utility, which I discuss in depth in this introduction, and throughout this thesis.

In the remainder of this section, I first review ideas from general relativity that are relevant to this thesis and provide motivation for the chapters to come: symmetries and conservation laws, black holes, and gravitational waves. I then provide a summary of the topics covered in this thesis.

1.1 Symmetries and Conservation Laws

In classical mechanics, the notion that symmetries give rise to conservation laws is a very familiar one, and is generally called *Noether's theorem*. In this section, we will be reviewing how this idea extends to general relativity.

1.1.1 Isometries

In general relativity, the dynamical field is the metric of the spacetime, g_{ab}. The components of this metric, $g_{\mu\nu}$ [2], are generally a function of the point (x^0, x^1, x^2, x^3) at which they are evaluated. As such, a natural notion of a symmetry is the requirement that components of the metric are independent of a particular coordinate; for example,

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0. \tag{1.3}$$

This type of symmetry is known as an *isometry*.

There are a variety of examples of isometries in solutions to the Einstein equations. An example that is particularly relevant is the Schwarzschild metric, which (as we will discuss in section 1.2), describes a black hole of mass M:

$$\mathrm{d}s^2 \equiv g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{1.4}$$

The components of this metric are clearly independent of both t and ϕ. Moreover, there are additional isometries, rotations about the x or y axes, which are not evident from equation (1.4), but can be seen in rotated coordinate systems. As such, this metric is fully *spherically symmetric*, being invariant under all spatial rotations.

[2] In this thesis, we use Latin letters from the beginning of the alphabet (a, b, etc.) to denote *abstract indices* [176], while we (generally) use Greek letters (μ, ν, etc.) to refer to *spacetime component indices*. As such, g_{ab} refers to metric as a geometrical object (a function of two vectors that gives the dot product between them), whereas $g_{\mu\nu}$, with $\mu, \nu = 0, 1, 2, 3$ gives the components of the metric in a specific coordinate system. Exceptions to this general rule will be explained when they arise.

At this point, it is useful to consider how symmetries can be understood in terms of vector fields on spacetime. Consider an infinitesimal change of coordinates of the form

$$x^\mu \to x^\mu + \epsilon \xi^\mu, \tag{1.5}$$

where ξ^μ are the components of a vector field and ϵ is a small parameter. Under such a change in coordinates, the metric changes by

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon(\xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + O(\epsilon^2), \tag{1.6}$$

where $\partial_\mu = \partial/\partial x^\mu$ denotes the usual coordinate derivative. The particular combination of terms at linear order in ϵ in this expression is a tensor, which is known as the *Lie derivative* $\pounds_\xi g_{ab}$ of the metric g_{ab} (for a more geometric definition of the Lie derivative, see appendix C of [176]). Vector fields that satisfy Killing's equation,[3]

$$\pounds_\xi g_{ab} = 2\nabla_{(a}\xi_{b)} = 0, \tag{1.7}$$

known as *Killing vectors*, are directly associated with isometries of a spacetime.

It is in terms of these Killing vectors that one can show that point particles in general relativity possess conserved quantities associated with isometries. A point particle in general relativity follows a path $\gamma(\tau)$, where τ is proper time, and the four-velocity of such a particle is simply the tangent to this curve, which we will denote by $\dot{\gamma}^a$. When this point particle is freely falling, the curve γ is a geodesic: between any two points along this curve, the total length of this curve is an extremum. This condition that the path length is extremized is equivalent to the *geodesic equation*:

$$\dot{\gamma}^b \nabla_b \dot{\gamma}^a = 0. \tag{1.8}$$

For a given Killing vector ξ^a, consider the quantity

$$E_\xi \equiv p^a \xi_a, \tag{1.9}$$

[3]Here, ∇_a is the covariant derivative, and we use the usual notation of symmetrizing over indices using round brackets (antisymmetrization will be denoted with square brackets).

where $p^a = m\dot{\gamma}^a$ is the momentum of the particle, and the mass m is constant. This quantity is conserved along γ, as[4]

$$\frac{\mathrm{d}E_\xi}{\mathrm{d}\tau} \equiv \dot{\gamma}^a \nabla_a E_\xi = m\left(\xi_b \dot{\gamma}^a \nabla_a \dot{\gamma}^b + \dot{\gamma}^a \dot{\gamma}^b \nabla_a \xi_b\right) = 0, \tag{1.10}$$

where the first term vanishes due to the geodesic equation (1.8) and the second due to Killing's equation (1.7).

1.1.2 Hidden symmetries

Not all conserved quantities, however, are related to explicit symmetries of the metric. Such conserved quantities are said to be related to so-called "hidden symmetries" of the spacetime. Essentially, the existence of hidden symmetries is due to the fact that Noether's theorem only gives a necessary, but not sufficient, condition for conserved quantities to exist.

There are many examples of such hidden symmetries, even in classical mechanics, where the analogous statement is that not all conserved quantities are related to symmetries of the Lagrangian. Perhaps the most famous example arises in the Kepler problem of planetary orbits: the Laplace-Runge-Lenz vector $\vec{e}$, given by (see, for example, section 3–9 of [79])

$$\vec{e} = \frac{\vec{v} \times (\vec{r} \times \vec{v})}{M} - \frac{\vec{r}}{r}, \tag{1.11}$$

where $\vec{r}$ and $\vec{v}$ are the position and velocity of the orbiting body (respectively) and M is the mass of the body that is being orbited (note that, as we have done throughout this thesis, we have set $G = 1$). One can show that this vector is constant in time, but there is no symmetry of the Lagrangian that directly gives rise to this conserved quantity. Moreover, this vector is conserved only for central forces that follow an inverse square law.

The reason why these conservation laws are said to be related to "hidden symmetries" is related to a particular feature of the Laplace-Runge-Lenz vector. There is a way of recasting the Kepler problem as a problem in one higher dimension [74], in which *additional* rotational symmetries arise. These rotational symmetries directly give rise to the conservation of the Laplace-Runge-Lenz vector.

[4] Here, $\mathrm{d}/\mathrm{d}\tau$ denotes the covariant derivative $\dot{\gamma}^a \nabla_a$ of a scalar along γ; later, we use $\mathrm{D}/\mathrm{d}\tau$ to denote the covariant derivative $\dot{\gamma}^a \nabla_a$ of any tensor along γ. The distinction made in this notation is due to the fact that $\mathrm{d}/\mathrm{d}\tau$ does not depend on the particular choice of covariant derivative ∇_a, whereas $\mathrm{D}/\mathrm{d}\tau$ does.

As such, the symmetry is present, but "hidden". However, not *all* conserved quantities are known to be related to symmetries that exist in a different formulation of a problem.

To show how a hidden symmetry can arise in general relativity, let us consider the following problem: we would like to define a generalization of a point particle's angular momentum in general relativity, in the sense that we want a vector that is linear in the momentum p^a of the particle and conserved for the case of freely-falling particles. Linearity in momentum implies that we can write

$$L^a = f^a{}_b p^b, \tag{1.12}$$

for some tensor f_{ab}. We now wish to determine whether L^a is conserved. Note that

$$\frac{\mathrm{D}L^a}{\mathrm{d}\tau} = \dot{\gamma}^b \nabla_b L^a = m\dot{\gamma}^b \dot{\gamma}^c \nabla_b f^a{}_c. \tag{1.13}$$

Since equation (1.13) should hold for all geodesics γ, we obtain

$$\nabla_{(b} f^a{}_{c)} = 0. \tag{1.14}$$

We now make the following observation: in classical mechanics, angular momentum arose as $\vec{r} \times \vec{p}$. Writing this expression out in coordinates, we find that the linear map that acts upon $\vec{p}$ is an antisymmetric matrix. Therefore, we also require that f_{ab} be antisymmetric, and arrive at the *Killing-Yano equation* (see, for example, section 35.3 of [151]):

$$\nabla_{(a} f_{b)c} = 0, \tag{1.15}$$

where f_{ab} is antisymmetric. Solutions to this partial differential equation are known as Killing-Yano tensors. There exist spacetimes in general relativity with non-trivial Killing-Yano tensors.

Similarly, one can think about generalizing the notion of "squared angular momentum" L^2 to general relativity. This conserved quantity is bilinear in the momentum, and so its generalization (which we will call K) must be of the form

$$K = K_{ab} p^a p^b, \tag{1.16}$$

where K_{ab} is symmetric. The conservation of K gives that

$$0 = \frac{\mathrm{d}K}{\mathrm{d}\tau} = m^2 \dot{\gamma}^a \dot{\gamma}^b \dot{\gamma}^c \nabla_a K_{bc}, \tag{1.17}$$

and so K_{ab} must satisfy the *Killing tensor equation* (see, for example, section 35.3 of [151]):

$$\nabla_{(a}K_{bc)} = 0. \tag{1.18}$$

Again, there are many examples of spacetimes in general relativity that possess Killing tensors. Moreover, for any spacetime that possesses a Killing-Yano tensor f_{ab}, one can show that $f_{ac}f^{c}{}_{b}$ is a Killing tensor.

The conserved quantities discussed in this section give a variety of useful results for solving problems. For example, the conservation of the Laplace-Runge-Lenz vector in the Kepler problem implies that the orbits are closed, and given by conic sections (with the magnitude of this vector giving the eccentricity). Another example is the class of spacetimes which describe spinning black holes, where a Killing tensor exists [179], and the associated conserved quantity allows for the solution of the geodesic equation in terms of first integrals [48]. This conserved quantity, the Carter constant, is a generalization of total angular momentum, and we discuss it extensively in chapters 2 and 3.

1.1.3 Conserved quantities for field theories on a fixed background

One may ask about the existence of conserved quantities for classical field theories in general relativity, such as electromagnetism. In electromagnetism on a flat background, the electromagnetic field stores energy, momentum, and angular momentum. Let us briefly review the sense in which this holds: for energy, for example, there exist quantities

$$U = \frac{1}{8\pi}(E^2 + B^2), \qquad \vec{S} = \frac{1}{4\pi}\vec{E}\times\vec{B}, \tag{1.19}$$

such that (in the absence of charged matter)

$$\frac{\partial U}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0. \tag{1.20}$$

This is a type of "continuity equation", much like that for the charge and current density:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0. \tag{1.21}$$

The integral form of the continuity equation, in the case of charge and current density, says that the change in the total charge of a system is entirely due to the flow of current through this system's boundary. The interpretation of equation (1.20) is then that the change in total energy stored in the electromagnetic fields is entirely due to a "flow of energy" through the boundary of the system.

In special relativity, the continuity equation for electric charge can be written in the following form:

$$\nabla_a j^a = 0, \tag{1.22}$$

where j^a is the four-vector *source current* $(\rho, \vec{J})$. A vector field j^a satisfying equation (1.22) is said to be conserved, and such a vector field is known as a *conserved current*. The energy U and Poynting vector $\vec{S}$ are components of the electromagnetic stress-energy tensor T^{ab}_{EM}, which is conserved in the sense that

$$\nabla_a T^{ab}_{\mathrm{EM}} = 0, \tag{1.23}$$

assuming no charged matter. In general, all non-gravitational fields possess a *total* stress-energy tensor T^{ab} that satisfies

$$\nabla_a T^{ab} = 0. \tag{1.24}$$

For any Killing vector ξ^a, one can define

$$j^a_{\mathrm{T}}[\boldsymbol{\xi}] \equiv T^{ab}\xi_b, \tag{1.25}$$

which is always conserved:

$$\nabla_a j^a_{\mathrm{T}}[\boldsymbol{\xi}] = \xi^b \nabla_a T^{ab} + T^{ab}\nabla_a \xi_b = 0, \tag{1.26}$$

where the first term vanishes by equation (1.24), and the second by Killing's equation (1.7). In electromagnetism, one can show that $(U, \vec{S})$ is a four-vector of the form (1.25), where $T^{ab} = T^{ab}_{\mathrm{EM}}$ and ξ^a is the Killing vector field corresponding to time translations.

In particular, in the case where there is both charged matter and electromagnetism, while $T^{ab}_{\mathrm{matter}}\xi_b$ and $T^{ab}_{\mathrm{EM}}\xi_b$ are not individually conserved, their sum is. This is a generalization of Poynting's theorem: in electromagnetism in flat spacetime, in the presence of sources, it is no longer true that equation (1.20) holds. Assuming that no particles leave a volume V, one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_{\mathrm{matter}} + \int_V U \mathrm{d}V\right) + \oint_{\partial V} \vec{S} \cdot \mathrm{d}\vec{A} = 0; \tag{1.27}$$

that is, the change in the total energy of the system is only given by the flow of energy stored in the electromagnetic fields.

In this section, we have constructed conserved currents associated with spacetime symmetries using the stress-energy tensor. There are a variety of other means of constructing conserved currents, such as the Noether current of [104] or the symplectic current of (for example) [44]. In chapters 3 and 5, we consider conserved currents that arise most naturally *without* the use of the stress-energy tensor, and we moreover show in chapter 2 that certain conserved currents *cannot* be constructed in such a way. In particular, we consider conserved quantities which are associated with the hidden symmetries discussed above, which cannot be understood naturally in terms of the stress-energy tensor.

1.1.4 Asymptotic symmetries

General spacetimes do not possess any isometries. However, if they represent an isolated gravitational system where the metric "approaches that of flat spacetime" sufficiently quickly, then they will possess *asymptotic* symmetries. These asymptotic symmetries give rise to interesting conserved quantities which we discuss below.

In order to understand the general picture of asymptotic flatness, it helps to consider how the metric behaves in flat spacetime. Consider the Minkowski metric in spherical coordinates:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{1.28}$$

We wish to consider this metric at large r, but at fixed retarded time $u = t - r$, motivated perhaps by a desire to study radiation far from an isolated source. However, the components of this metric blow up as $r \to \infty$, specifically in the angular directions. Another issue that arises in this discussion is that we are taking a limit to infinity—it more convenient in calculations to instead consider a limit of some coordinate to a finite value, and so we define $\Omega \equiv 1/r$, and use Ω instead of r as a coordinate.[5] Performing a coordinate transformation to $(u, \Omega, \theta, \phi)$, we find that

$$\begin{aligned} \mathrm{d}s^2 &= \Omega^{-2}\left(-\Omega^2 \mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}\Omega + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \\ &\equiv \Omega^{-2}\mathrm{d}\tilde{s}^2. \end{aligned} \tag{1.29}$$

The metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ associated with the interval $\mathrm{d}\tilde{s}^2$ is a conformal transformation of the physical metric g_{ab}, and is called the "unphysical metric". The components of the unphysical metric are now

[5] We are considering the limit of $r \to \infty$, so we do not worry about the poor behavior of Ω at $r = 0$; for a definition of Ω that works throughout the Minkowski spacetime, see, for example, the discussion in section 11.1 of [176]).

finite in the limit $\Omega \to 0$:

$$\mathrm{d}\tilde{s}^2\big|_{\Omega=0} = 2\mathrm{d}u\mathrm{d}\Omega + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2. \tag{1.30}$$

The surface $\Omega = 0$, which is not a part of the physical spacetime (as it corresponds to "$r = \infty$"), is known as *future null infinity*, and is denoted by the symbol $\mathscr{I}^+$ (pronounced "scri plus"). There is a similar surface $\mathscr{I}^-$, *past* null infinity, which is constructed in an analogous way, but by taking a limit $r \to \infty$ at fixed $v = t + r$. Both surfaces is null, much like the light-cone in flat spacetime. Many of the unintuitive features of null infinity are due to the fact that it is a null surface, as much of our intuition about surfaces in flat three-dimensional Euclidean space fails to hold.

This construction of null infinity generalizes to a much larger class of spacetimes, which are called *asymptotically flat* (for an extensive review, see [77]). That is, for a large class of spacetimes, there exists an unphysical metric $\tilde{g}_{ab}$ related to the physical metric g_{ab} by

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \tag{1.31}$$

where the conformal factor $\Omega > 0$ in the physical spacetime. The physical spacetime can then be extended to larger, *unphysical* spacetime, possessing a boundary $\mathscr{I}$ where Ω vanishes. Moreover, it can be shown (under more technical assumptions [77]) that the intrinsic geometry of $\mathscr{I}$ for all asymptotically flat spacetimes is the same as that for the Minkowski spacetime, with all metrics at $\Omega = 0$ able to be mapped to the form (1.30). This is the sense in which these spacetimes are "asymptotically flat"; one can, moreover, show that the physical metrics of asymptotically flat spacetimes agree with that of the Minkowski spacetime, up to terms of order $1/r$.

Since asymptotically flat spacetimes are asymptotically close to Minkowski spacetime, one would expect that the asymptotic symmetry group of such a spacetime should be the same as that of Minkowski: the ten-dimensional Poincaré group. Although it is by no means obvious, this is not the case: the asymptotic symmetry group is instead the *infinite-dimensional* Bondi-Metzner-Sachs (BMS) group [37, 143]. A general member of this group maps the coordinates (u, θ, ϕ) of $\mathscr{I}^+$ as follows. The angular coordinates are mapped according to

$$\theta \to H(\theta, \phi), \qquad \phi \to I(\theta, \phi), \tag{1.32}$$

which acts as a conformal isometry of the sphere,

$$\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 \to [K(\theta, \phi)]^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{1.33}$$

(that is, the sphere is mapped to itself, up to scaling). Similarly, u is mapped via

$$u \to [K(\theta,\phi)]^{-1}[u + \alpha(\theta,\phi)]. \tag{1.34}$$

We now discuss the various pieces of this group. Conformal isometries of the sphere can be understood in terms of conformal mappings in the complex plane, and such an analysis shows the space of conformal isometries is six-dimensional. Three of these conformal isometries are pure isometries $[K(\theta,\phi) = 1]$ corresponding to rotations, and the remaining three correspond to boosts; in total, they give the six-dimensional Lorentz group. The infinite-dimensional portion of the BMS group comes from considering transformations where $H(\theta,\phi) = \theta$, $I(\theta,\phi) = \phi$ [which implies that $K(\theta,\phi) = 1$]:

$$u \to u + \alpha(\theta,\phi). \tag{1.35}$$

Such a transformation, where α is an arbitrary function of θ and ϕ, is known as a *supertranslation*. The relationship to translations can be understood as follows: in flat spacetime, performing a translation $x^\mu \to x^\mu + \Delta x^\mu$ results in the following leading-order change in u:

$$u \to u + \Delta t - \sin\theta\cos\phi\,\Delta x - \sin\theta\sin\phi\,\Delta y - \cos\theta\,\Delta z + O(1/r). \tag{1.36}$$

The coefficients in front of Δt and Δx, Δy, Δz are, respectively, proportional to the $l = 0$ and $l = 1$ spherical harmonics. The translation (1.36) is of the form (1.35), with $\alpha(\theta,\phi)$ being a sum of $l < 2$ spherical harmonics, and therefore supertranslations can be thought of as a generalization to the usual translation group, including all $l \geq 2$ spherical harmonics.

In the Poincaré group, there is no unique Lorentz subgroup. Instead, for each choice of origin in Minkowski spacetime, there exists a subgroup of Lorentz transformations that keep that origin fixed. These origins are, of course, related by translations. Similarly, the BMS group has an *infinite-parameter family* of such Lorentz subgroups, the "origins" being related by supertranslations. The existence of supertranslations is also related to gravitational waves, in the following sense: the action of a linearized supertranslation on the Minkowski spacetime produces a change in the metric that falls off near null infinity at the same rate as a gravitational wave, and so (in this sense) they are indistinguishable.

With the discussion of asymptotic symmetries concluded, we now turn to conservation laws. Here, like in the case of point particles, the conserved quantities are simply numbers that are associated with a given symmetry (in this case, a member of the BMS group); for example, the total mass of a spacetime is associated with an asymptotic time translation [constant α in equation (1.35)]. However, these "conserved quantities" are only conserved in the absence of radiation—one would expect the mass, for example, of a spacetime to decrease with the radiation of gravitational waves. As such, these conserved quantities should be associated with particular values of retarded time: in fact, they are given by integrals over cross-sections of null infinity, and such cross-sections are labeled by a particular value of retarded time. Such an integral over a spherical cross-section at infinity that is associated with a symmetry is known as a *charge*. The name arises due to the nature of the total electric charge in electromagnetism, which is given by a surface integral of the electric field over a sphere at infinity, and is related to the existence of gauge symmetry.

One might now ask how to determine the charges that are associated with the members of the BMS group. One approach is similar to that taken in electromagnetism: the total electric charge Q is given by a certain component of the electric field, in the limit of large r:

$$E_r = \frac{Q}{4\pi r^2} + O(1/r^3). \tag{1.37}$$

Similarly, one can show that [37]

$$g_{uu} = -1 + \frac{2M(u,\theta)}{r} + O(1/r^2). \tag{1.38}$$

The $M(u,\theta)$ is related to the so-called "Bondi mass" $M(u)$ of the system by

$$M(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta\,\mathrm{d}\theta. \tag{1.39}$$

The Bondi mass equals the mass in stationary spacetimes. Similar expressions can be used to read off angular momentum from components of the metric. However, it is not clear how these expressions generalize to conserved quantities related to a general asymptotic symmetry.

A very effective method of constructing conserved charges in general relativity is the *Wald-Zoupas procedure* [178]. This procedure is essentially a version of Noether's theorem for radiating systems: given any theory characterized by a Lagrangian and an asymptotic symmetry group, it provides a method of constructing charges associated with each asymptotic symmetry. Moreover, the change in such a charge, which are known as the *flux*, is given by an integral over null infinity of a current which only depends on the radiative degrees of freedom of a theory. In chapter 5, we give a more thorough overview of the Wald-Zoupas procedure, applying it to the theory of electromagnetism coupled to general relativity.

1.2 Black Holes

Black holes are one of general relativity's stranger predictions, and also one of its first: the Schwarzschild metric (1.4), which describes a spherically symmetric black hole, was discovered in 1916 [146]. Black hole solutions are characterized by the existence of horizons, boundaries in spacetime through which objects can only pass one way. While these solutions were originally thought to be unphysical, merely products of exact spherical symmetry, it was eventually concluded that gravitational collapse should generically produce black holes, especially with the discovery of axisymmetric black holes in the form of the Kerr solution [108] and the singularity theorems (for a review, see chapter 8 of [93]).

The initial astrophysical evidence for black holes was the existence of X-ray binaries: binary systems where gas from a star accretes onto an unseen companion (for a review, see, for example, chapter 13 of [149]). While the mass of this unseen companion can be inferred, and shown to be more massive than allowed by compact objects that are not black holes, this form of detection is indirect. The first direct evidence as to the existence of black holes came from the detection of gravitational waves by the Laser Interferometer Gravitational-wave Observatory (LIGO) in 2015 [3]. These gravitational waves were produced by the merger of a binary black hole system, as confirmed by comparison with numerical models of such a system. Further binary black hole mergers have also been detected; for a catalog of the detections, see [6]. I will review LIGO in more depth in section 1.3.2. In addition to gravitational wave detection, there is now electromagnetic evidence of black holes: the Event Horizon Telescope, a global system of radio telescopes, has imaged the radio emission surrounding the supermassive black hole at the center of the galaxy M87 [8]. The "shadow" at the center of this image is consistent with the existence of an event horizon, although it is of course larger than the event horizon due to the gravitational deflection of light.

In the remainder of this section, I discuss in further detail the nature of black hole horizons and the Kerr spacetime describing a spinning black hole, both of which are relevant to chapters 2 and 3.

1.2.1 Horizons

The one defining feature of a black hole is its event horizon: a boundary in spacetime through which objects can enter, but not return. In this section, I will briefly review why, intuitively, this occurs.

As a simple example, we will consider the event horizon of the Schwarzschild solution in equation (1.4). This metric appears to be singular at the point $r = 2M$, but such a singularity is merely a coordinate artifact: using a sort of advanced time coordinate v,

$$v = t + r^*, \tag{1.40}$$

where r^* is the *tortoise coordinate* defined by

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{1}{1 - 2M/r}, \tag{1.41}$$

one finds that the metric in equation (1.4) takes the form

$$\mathrm{d}s^2 = -(1 - 2M/r)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{1.42}$$

In this coordinate system, it is apparent that there is no issue at the surface $r = 2M$, which is in fact a null surface, like null infinity above in section 1.1.4, or a light-cone in the Minkowski spacetime. Once one has entered the future light-cone of some point, one cannot exit it—this is precisely why the event horizon functions as a one-way boundary. It should be noted, however, that horizons are unlike lightcones, in the following sense: while the interior of the future lightcone of a point extends to infinity, the interior of the horizon only includes points with $r < 2M$. As such, no particle can reach $r = \infty$ once it has passed through the event horizon.

One minor note is that this is a *future* horizon, acting like the the future light-cone of a point in Minkowski space. By using a coordinate $u = t - r^*$, one can construct from the metric 1.4 a *past* horizon, which act like the past light-cone, only allowing a particle to move into the region $r > 2M$. The existence of a past horizon for the Schwarzschild spacetime is a consequence of the fact that the metric (1.4) describes an *eternal* black hole—the components of the metric are independent of time, and so the black hole has always existed. For physical black holes which are formed by gravitational collapse, this past horizon does not exist.

In chapter 3, we will consider fluxes of conserved currents through the horizon of a black hole. These are similar to fluxes of conserved currents through null infinity: much as the flux of energy (for example) at null infinity represents energy lost to infinity, the flux through the horizon represents energy lost to the black hole.

1.2.2 The Kerr spacetime

A particular example of a black hole spacetime that is relevant to this thesis is the Kerr spacetime, which represents a black hole with both mass M and spin Ma [108]. As shown in [46, 92, 139], this metric represents (under certain assumptions) the unique stationary black hole solution in vacuum general relativity, and so is expected to be the final state of gravitational collapse. In particular, this metric is characterized by only two degrees of freedom, the mass and spin. All higher multipole moments are essentially trivial, and determined solely from these two numbers; note that this is *not* the case for the matter from which these black holes formed! This lack of features of a black hole is known as the "no-hair theorem", and is an important result of general relativity that gravitational wave astronomy hopes to verify.

We describe in much greater details the features of the Kerr spacetime in chapters 2 and 3. However, a particular feature that is relevant to mention at this point, as motivation for further discussion in those chapters, is the existence of an additional constant of motion for point particles, the *Carter constant*, which does not arise due to the two isometries, t- and ϕ-independence. We will briefly review how this constant arises in section 2.1. This constant, which we denote by K, is bilinear in the momentum of the particle, and so, as mentioned in section 1.1.2, it is associated with a Killing tensor K_{ab}:

$$K = K_{ab} p^a p^b. \tag{1.43}$$

This constant is also a generalization of the total squared angular momentum L^2 to the Kerr spacetime, as we show in section 2.1 by a specialization to the Schwarzschild spacetime. The Kerr spacetime moreover possesses a Killing-Yano tensor f_{ab} such that

$$K_{ab} = f_{ac} f^c{}_b. \tag{1.44}$$

The properties of this Killing-Yano tensor will be discussed further in chapter 3.

1.3 Gravitational Waves

The existence of propagating gravitational waves has been predicted since almost the very beginning of the history of general relativity, with Einstein's first paper on the subject in 1916 [61]. The reality of gravitational waves, however, was not generally agreed upon until the 1957 Chapel Hill conference [30], where a thought experiment was proposed suggesting that gravitational waves carried energy, as they could move a "sticky bead" along a wire (later published in [35]).

The ability to directly detect a gravitational wave has only been achieved recently. The first (indirect) detection came from the Hulse-Taylor pulsar, the first pulsar to be found in a binary system [100]. Once the masses of both the pulsar and its unseen companion were determined, the measured rate of decay of the orbital period could be compared with the prediction of loss of energy due to gravitational waves [161]. On the other hand, direct gravitational wave detection was initially met with less success, with the first claimed detections by means of a resonant bar [182] found to be irreproducible (for a review, see [114]).

Perhaps the greatest achievement in the field of general relativity in recent years was the detection of gravitational waves from the the Laser Interferometer Gravitational wave Observatory (LIGO) in 2015 [3]. This observatory, composed of two 4-km interferometers built in Hanford, Washington and Livingston, Louisiana, was initially constructed in 1994, achieving initial design sensitivity in 2005 [2]. While initial science runs yielded no detections, the completion of upgrades to so-called "Advanced LIGO" in 2014 increased the sensitivity by a factor of ten [1], allowing for the initial detections in 2015. The first source of these gravitational waves were the collision of two black holes of nearly equal masses, $35.6^{+4.7}_{-3.1}\,M_\odot$ and $30.6^{+3.0}_{-4.4}\,M_\odot$. Since this initial detection, there have been many detections of both binary black hole mergers (for a catalog, see [6]), as well as a binary neutron star merger [4]. In particular, the first detected binary neutron star merger was accompanied by a gamma ray burst, and was subsequently followed up in detections across the electromagnetic spectrum [5]. This detection heralded the start of the era of *multi-messenger astronomy*, where events can be detected both in electromagnetic and gravitational waves, each providing information that is complementary to the other.

In the remainder of this section, I will review the basic theory of gravitational waves and their detection. Moreover, I briefly review the sources of gravitational waves relevant to this thesis, in the form of two physical phenomena that have not yet been observed: extreme mass-ratio inspirals (EMRIs) and the gravitational wave memory effect. For a review of the formalism in this section, see (for example) [71].

1.3.1 Basic formalism

The theory of gravitational waves is (typically) considered in the regime of linearized gravity on a background spacetime, which we review in this section. In particular, the background spacetime is typically assumed to be the flat Minkowski spacetime, although that is not strictly necessary for much of the formalism. In section 4.3, we consider exact solutions to the Einstein equations which possess many of the properties of these linearized solutions; they are known as *nonlinear plane wave spacetimes*. Such spacetimes are relevant for considering corrections to the linearized theory that may arise when considering (for example) the long-time behavior of gravitational waves.

The basis of linearized gravity is the following: the full metric is assumed to a member of a one-parameter family $g_{ab}(\lambda)$ that is of the form

$$g_{ab}(\lambda) = g_{ab} + \lambda \delta g_{ab} + \frac{\lambda^2}{2}\delta^2 g_{ab} + \cdots , \tag{1.45}$$

where g_{ab} denotes the background metric; in the rest of this section, we will assume that the background metric is flat: $g_{ab} = \eta_{ab}$. The notation "δg_{ab}" here is reminiscent of the fact that a variation in the variational formalism is a linearization about a background solution to the equations of motion; typically, the linearized metric is denoted h_{ab}, although one quickly runs out of letters if considering higher order terms in the series (1.45).

Note that there may be regimes in which the leading-order non-background term $\lambda \delta g_{ab}$ is not sufficient to physically describe the behavior of the full solution $g_{ab}(\lambda)$. Another complication is that equation (1.45) explicitly separates the full metric $g_{ab}(\lambda)$ into "background" and "perturbation", which is not unique [we are only given $g_{ab}(\lambda)$ at a specific value of λ, from which we are constructing the series (1.45)]. However, there *is* a sense in which a background and a perturbation can be separated: if g_{ab} is slowly-varying over some background scale L, and δg_{ab} (as well as higher order terms) vary over some lengthscale ℓ, then g_{ab} and δg_{ab} can be distinguished if $\ell \ll L$.

These complications aside, one can insert the solution $g_{ab}(\lambda)$ into the Einstein equations and solve these equations at each order in λ. For simplicity, we assume that there is no matter present, and so the Einstein equations can be simply written as $R_{ab}(\lambda) = 0$. At first order, one finds that

$$\Box \delta g_{ab} + \nabla_a \nabla_b \left(\eta^{cd} \delta g_{cd} \right) - 2\nabla_{(a} \nabla^c \delta g_{b)c} = 0, \tag{1.46}$$

where ∇_a is the covariant derivative compatible with the flat metric η_{ab}, and $\Box$ the usual d'Alembertian $\eta^{ab}\nabla_a\nabla_b$.

The key insight here, and the reason why these are gravitational *waves*, is that this equation is *almost* in the form of a wave equation. As in the case of electromagnetism, it can be put exactly in the form of a wave equation by using the gauge freedom that is present. In general relativity, there is always the freedom to change coordinates. Infinitesimal changes in coordinates, as we saw in equation (1.5), can be characterized by a vector field ξ^a. One can show that this coordinate freedom translates into a sort of gauge freedom for the metric perturbation δg_{ab}:

$$\delta g_{ab} \to \delta g_{ab} + 2\nabla_{(a}\xi_{b)}. \tag{1.47}$$

One can easily show that this gauge freedom can be used to impose Lorenz gauge, $\nabla^a \delta g_{ab} = 0$. Moreover, one can simultaneously set the trace $\eta^{ab}\delta g_{ab}$ to zero. In this particular gauge, one finds that

$$\Box \delta g_{ab} = 0, \tag{1.48}$$

which is *exactly* a wave equation.

There are additional gauge transformations that can be performed, specifically assuming that there are no sources. In this case, one can further restrict to *transverse traceless (TT) gauge*, where

$$\delta g_{ab} = h^{\mathrm{TT}}_{ab}, \tag{1.49}$$

with the tensor on the right-hand side of this equation satisfying[6]

$$h^{\mathrm{TT}}_{0\mu} = 0, \qquad \delta^{ij} h^{\mathrm{TT}}_{ij} = 0, \qquad \nabla^j h^{\mathrm{TT}}_{ij} = 0. \tag{1.50}$$

[6] Here, spatial indices are denoted with Latin letters from the middle of the alphabet (i, j, etc.), and take on the values $1, 2, 3$.

The purely spatial components h_{ij}^{TT} only have

$$6 \text{ (symmetric)} - 1 \text{ (trace-free)} - 3 \text{ (divergence-free)} = 2 \tag{1.51}$$

degrees of freedom; in the case where h_{ij}^{TT} is purely a function of $t - z$, say (a gravitational wave propagating in the z direction), one therefore has that

$$h_{ij}^{\mathrm{TT}}(t, x, y, z) = h_{+}(t - z)e_{ij}^{+} + h_{\times}(t - z)e_{ij}^{\times}, \tag{1.52}$$

where

$$e_{11}^{+} = -e_{22}^{+} = 1, \qquad e_{12}^{\times} = e_{21}^{\times} = 1, \tag{1.53}$$

with all other components vanishing. This choice of gauge confirms that, like electromagnetism, linearized gravity possesses only two radiative degrees of freedom corresponding to two polarizations, which are called "+" and "×", respectively.

1.3.2 Detection

In this section, we discuss the detection of gravitational waves. We begin with a brief overview of the theory of gravitational wave detection, and then review the current and future gravitational wave detectors that are relevant for the topics considered in this thesis.

1.3.2.1 Formalism

The basis for gravitational wave detection is the accurate measurement of distance between two separated test masses. There are two ways of thinking about how this process works. The first, which is often given in more elementary discussions of gravitational waves, is the following: the metric in the TT-gauge coordinate system is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(\delta_{ij} + \lambda h_{ij}^{\mathrm{TT}}\right) \mathrm{d}x^i \mathrm{d}x^j. \tag{1.54}$$

Consider the proper separation ξ^i between two closely-separated points; much like proper time, this is the distance (as measured using the metric) between these two points (in this case, the two points are spacelike, as opposed to timelike, separated). For such points with coordinate separation x^i, the proper separation is approximately

$$\xi_i \simeq \left(\delta_{ij} + \frac{1}{2}\lambda h_{ij}^{\mathrm{TT}}\right) x^j. \tag{1.55}$$

Now, it is easy to show (see, for example, [71]) that the coordinate separation of freely-falling point masses in the TT-gauge coordinate system is constant, assuming initially comoving observers and small velocities. While this initially appears to imply that there is no effect of the gravitational wave on these observers, equation (1.55) shows that the *proper separation* does, in fact, change. The difference between the initial and final proper separations is given by

$$\Delta\xi_i \simeq \frac{1}{2}\lambda \Delta h_{ij}^{\mathrm{TT}} \xi^j, \tag{1.56}$$

where $\Delta h_{ij}^{\mathrm{TT}}$ is the difference between the initial and final TT-gauge metric perturbation. As h_{ij}^{TT} is traceless, the effect of the passage of gravitational waves is a shear.

The above discussion, however, is somewhat limited in scope. In general, given two freely falling observers, the separation vector between these observers obeys the *geodesic deviation equation*:

$$\ddot{\xi}^a \simeq -R^a{}_{cbd}\dot{\gamma}^c\dot{\gamma}^d\xi^b, \tag{1.57}$$

where $\dot{\gamma}^a$ is the four-velocity of one of the observers, and the terms that are neglected are appropriate to neglect in the limit of small separation and relative velocity. We will discuss the derivation of this equation in more detail in section 4.2.2.1. One can show, moreover, that in TT-gauge,

$$\delta\!\!\!{}^{-} R_{i0j0} = -\frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}}, \tag{1.58}$$

where dots refer to derivatives with respect to time in this expression. Assuming that the observers are stationary in the TT-gauge coordinate system, then we recover that

$$\ddot{\xi}_i \simeq \frac{1}{2}\lambda \ddot{h}_{ij}^{\mathrm{TT}} \xi^j. \tag{1.59}$$

Integrating this equation twice in time directly gives equation (1.56).

1.3.2.2 Ground-based detectors

We begin with the a discussion of ground-based detectors, which (so far) are the only gravitational wave detectors to have successfully made detections. There are currently five operational gravitational wave detectors:

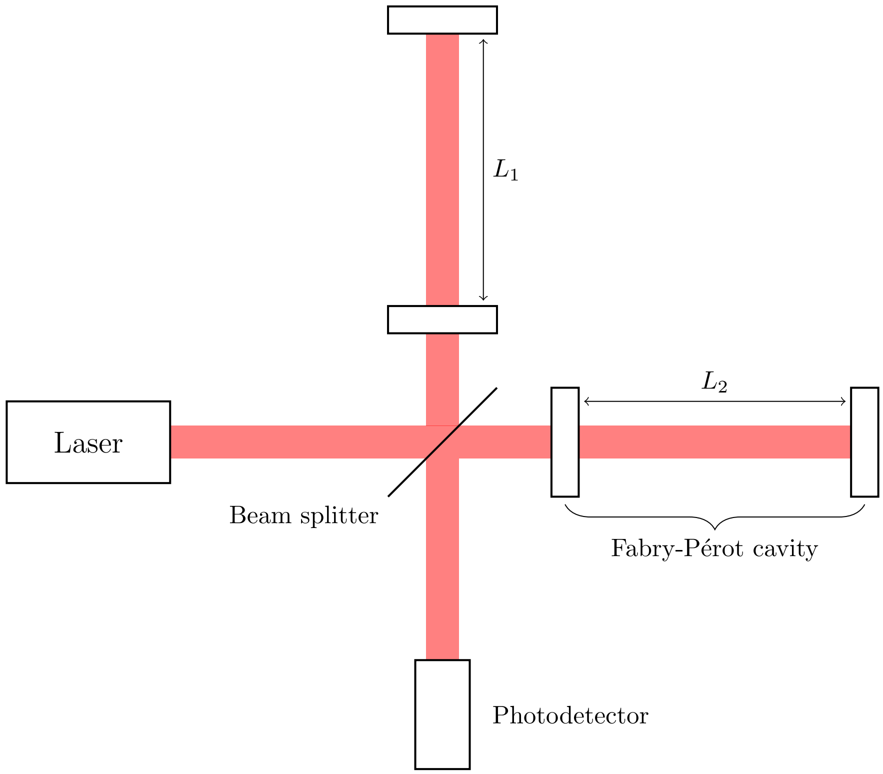

Figure 1.2: A Michelson interferometer of the sort used for current ground-based for current gravitational wave detection. The arms of this interferometer, of length L_1 and L_2, are both resonant Fabry-Pérot cavities in order to increase sensitivity.

- the two LIGO detectors in Livingston, Louisiana and Hanford, Washington [1],
- the Virgo detector near Pisa, Italy [7],
- GEO600, a detector near Hannover, Germany [59], and
- the Kamioka Gravitational Wave Detector (KAGRA) in Gifu Prefecture, Japan [24].

All current ground-based detectors are essentially Michelson interferometers; a simplified diagram of such an interferometer is given in figure 1.2. The basic function of these interferometers is as follows: light that is emitted by a laser is split by a central beam splitter, sending the light into the two "arms" of the interferometer, with lengths L_1 and L_2. In all ground-based detectors (other than GEO600), the light in each arm reflects between the two mirrors on either end, forming a resonant "Fabry-Pérot cavity". The presence of this resonant cavity greatly increases the sensitivity of the detector to changes in the length of the arms compared to a basic Michelson interferometer (such as

GEO600), where there is only one mirror at the end of each arm (for a pedagogical introduction to how this arises, see section 9.5 of [166]). The two beams which exit the arm are then recombined by the beam splitter, and the interference between the two signals (as measured by a photodetector) measure changes in the relative lengths of the two arms.

Note that the length of the arms that is being measured by the interferometer is not the coordinate separation, but the proper separation (the true length of the arms). The two mirrors at either end of each arm function as the point particles in the above discussion of geodesic deviation. Note that these mirrors are not freely-falling, but this does not qualitatively change the above discussion in horizontal directions. Moreover, the interferometer only measures the relative change in lengths of the two arms, not the length of each arm individually.

The primary sources of noise in these ground-based interferometers are the photon shot noise at high frequencies and the radiation pressure noise at low frequencies (see, for example, [165]). The shot noise is due to the fact that the phase shift measured by the interferometer is limited by the total number of photons that hit the mirrors (essentially it is limited by $1/\sqrt{N}$), and so scales inversely with the power of the laser. The radiation pressure noise is due to the fluctuation in the radiation pressure on the mirrors, and so scales with power. By changing the power of laser, it is therefore possible to adjust the trade-off between shot and radiation pressure noises.

There are two other sources of noise that are relevant to ground-based detectors. The first is thermal noise, which can be understood in terms of fluctuations in the length of the arms on the order of $\sqrt{kT/(m\omega^2)}$, where m is the mass of whatever is oscillating (for example, atoms in the mirrors), and ω its natural frequency. Note, however, this effect is both averaged over both the entire surface of the mirrors (and so the thermal oscillations of individual atoms are unimportant), and over the average time that a photon spends in the Fabry-Pérot cavity [165]. For more details on the dominant contribution to the thermal noise due to the coating of the mirrors, see [63].

At low frequencies, another very important source of noise is in seismic activity. In addition to vibrations passing through the suspension system, there is also so-called “gravitational gradient noise”, which comes from gravitational fluctuations due to density perturbations created from seismic waves (see, for example, [99]). Seismic noise is almost unavoidable at the surface of the earth, which motivates the transition to space-based detectors for lower-frequency signals.

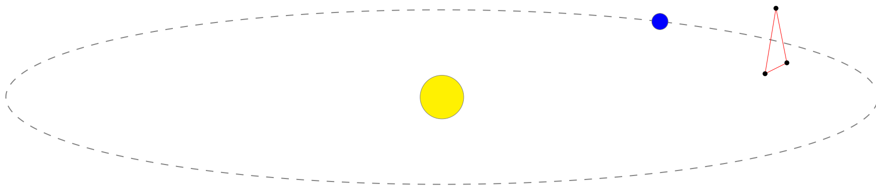

Figure 1.3: A depiction of the planned space-based LISA gravitational wave detector. This detector will be composed of three spacecraft orbiting the sun, trailing the earth, in a triangular formation, forming a system of three interferometers.

1.3.2.3 Space-based detectors

The Laser Interferometer Space Antenna (LISA) is a planned space-based detector composed of three spacecraft that orbit the sun in a near-equilateral triangle formation, resulting in a system of three interferometers with an arm-length of about 2.5×10^9 m [12], depcted in figure 1.3. Each spacecraft has two test masses, and sends out two laser beams to the two other craft, which are then transmitted back by phase-locked lasers. The test masses in the spacecraft are used to establish "drag-free" operation, whereby the test masses are freely falling and the spacecraft follows the test masses by determining any deviation in position of the test masses relative to the spacecraft (created by effects such as solar wind on the spacecraft, from which the test masses are shielded) and compensating with the spacecraft's thrusters. Using the phase information from all of the lasers, each spacecraft can determine any relative changes in the arm lengths, and so measure the passage of a gravitational wave.

As with ground-based detectors, the shot and radiation pressure noises will both be present in LISA, although LISA will be tuned such that its primary range of sensitivity will be at much lower frequency than ground-based detectors, since it can avoid the low-frequency seismic noise that are present in such detectors. Other than the shot and radiation pressure noise, there are other acceleration noises coming from (for example) thermal gradients in the spacecraft and electrostatic charging of the test masses by cosmic rays [12]. In addition to sources of noise in the detector itself, there is also an expected background of white dwarf binary systems that will limit the sensitivity of LISA in certain frequencies [142].

1.3.3 Sources

The discussion of the theory of gravitational waves in section 1.3.1 specialized entirely to the case where there are no sources, and one would like to understand how sources generate gravitational waves in order to predict the types of signals that one would expect. For slow-moving, weakly self-gravitating systems, one can simply add sources into the discussion in section 1.3.1. Far from a source, the spatial components of the metric in the above transverse-traceless gauge are given by the *quadrupole formula*:

$$h_{ij} = \frac{2}{r}\ddot{I}_{ij}^{\mathrm{TT}}(t-r), \tag{1.60}$$

where I_{ij}^{TT} is the reduced quadrupole tensor of the source, defined by:

$$I_{ij}^{\mathrm{TT}}(t) = \left(P_i{}^k P_j{}^l - \frac{1}{2}P_{ij}P^{kl}\right)\int_V \rho(t,\vec{x}\,')x'^i x'^j \mathrm{d}^3x', \tag{1.61}$$

and $P_{ij} = \delta_{ij} - x^i x^j/r^2$ is a projection operator. This equation is analogous to the expressions for the electromagnetic field far from a source in terms of dipole moment of the source. In cases where the self gravity of the system is not negligible, such as binary systems, one can show that the derivation of the quadrupole formula (1.60) still holds, so long as $\rho(t,\vec{x}\,')$ includes the gravitational binding energy of the source (see, for example [71]).

1.3.3.1 Extreme mass ratio inspirals (EMRIs)

The detection of binary mergers of black holes and neutron stars by LIGO has provided tests of the general relativistic two-body problem in the case of nearly equal masses. The characteristic frequency f_{merge} of gravitational waves at the merger of two objects is roughly the Keplerian orbital frequency at a separation on the order of the Schwarzschild radii of the two bodies:

$$\frac{1/f_{\mathrm{merge}}^2}{(m_1+m_2)^3} \sim \frac{1}{m_1+m_2}, \tag{1.62}$$

where m_1 and m_2 are the two masses, and so one finds that $f_{\mathrm{merge}} \sim 1/(m_1+m_2)$. As such, more massive objects tend to merge at lower frequencies, and so the LISA mission gives the opportunity to test the two-body problem in a fundamentally different regime, where the total mass is large, but the mass-ratio is not of order unity, and in fact small. These are extreme mass-ratio inspirals, or

EMRIs. They consist of a neutron star or stellar-mass black hole traveling in the gravitational field of a supermassive black hole (with mass $M \sim 10^7 - 10^{10} M_\odot$). Detecting these inspirals has both astrophysical consequences, in probing the population of stellar-mass objects near supermassive black holes, as well as physical applications, in testing general relativity by mapping the geometry around black holes and verifying the no-hair theorem described in section 1.2.2 (see, for example, [11]).

In the small mass-ratio limit, the objects orbiting the supermassive black hole can be well-modeled by point particles traveling in the Kerr spacetime. At lowest order in the mass ratio, we can consider these point particles to be following geodesics, but there clearly must be effects of the backreaction of their mass on the spacetime in which they are traveling and on their own motion. This is very similar to the *self-force* problem in electromagnetism (see, for example, the discussion in section 16.2 of [105]). While the force on a point particle due to its own gravitational field at its location is technically singular, a finite and universal self-force is obtained for extended bodies in the limit where their mass and size are taken to zero (see, for example, [80]). In this limit, the self-force naturally arises as the first leading-order correction to geodesic motion.

A particularly intuitive way of understanding the effects of the self force, at least in an averaged sense, is through so-called "flux-balance laws". In flat space, for example, an accelerating, charged particle creates an electromagnetic field with a non-zero flux of energy and momentum at infinity. The particle transfers energy and momentum to the electromagnetic fields near the particle (in the "near zone"), and these fields then transfer energy and momentum both back to the particle itself, and also to the fields far away (in the "far zone"). The energy and momentum in the far zone is transferred to infinity and lost forever; assuming that the near zone electromagnetic fields and the particle reach a steady state, the rate of loss of energy and momentum to infinity is, in an averaged sense, the self-force on the particle. A similar flux of angular momentum results in a self-torque.

The above paragraph describes the electromagnetic self-force in the case of flat spacetime, but similar ideas also hold in general relativity for the self-force problem in the Kerr spacetime (see, for example, [119, 145, 98]). However, these flux-balance laws only allow for computations of the *dissipative* self-force, which in the above picture comes from the loss of energy and momentum to infinity. The *conservative* self-force, which gives rise to oscillations in the orbits which are averaged out over long times, is not given by flux-balance laws. There are a wide array of computational strategies that have been employed to compute both the conservative and dissipative self-force in the Kerr spacetime (see [181] for a review). Recently, the full self-force in Kerr, to first order in perturbation theory, has been computed using the so-called "mode-sum regularization" method [171]. Even though flux-balance arguments do not give the full self-force, they have provided a useful check of results of first-order self-force calculations; second-order calculations are sufficiently complex that flux-balance arguments may provide crucial insight.

It is therefore interesting to consider flux-balance arguments in the Kerr spacetime. Here, however, there is not a ten-dimensional group of isometries, as there are in flat spacetime. In order to determine the behavior of a radiating point particle in the Kerr spacetime, one must know the evolution of its energy, angular momentum in the z direction (axial angular momentum), and its Carter constant. There are conserved currents associated with energy and axial angular momentum for gravitational waves, and these conserved currents can be used to compute the change in the energy and axial angular momentum of a particle. While the change of the Carter constant can be computed by other methods [119, 145, 103], the current status of conserved currents for the Carter constant is more complex, and will be discussed in chapters 2 and 3.

1.3.3.2 Gravitational wave memory

The gravitational wave memory effect has, historically, been described as the enduring relative displacement of two closely-separated observers that is produced by a passing burst of gravitational waves. Such a situation is depicted in figure 1.4. Zel'dovich and Polnarev [187] first noticed the effect in a calculation in linearized gravity of the fly-by of two astrophysical compact objects. The source of this effect can be readily seen from combining equation (1.56) and the quadrupole formula (1.60):

$$\Delta\xi_i \simeq \frac{2}{r}\Delta\ddot{I}_{ij}^{\mathrm{TT}}\xi^j. \tag{1.63}$$

Thus, if a source evolves from one state to another with differing second time derivative of the quadrupole tensor, such as can occur in the scattering of two massive objects, there will be a lingering effect on distant observers that does not go away once the gravitational wave has passed.

In addition to the gravitational scattering of compact objects, enduring displacements have been shown to occur in other astrophysical contexts. While there is no memory effect in the linearized

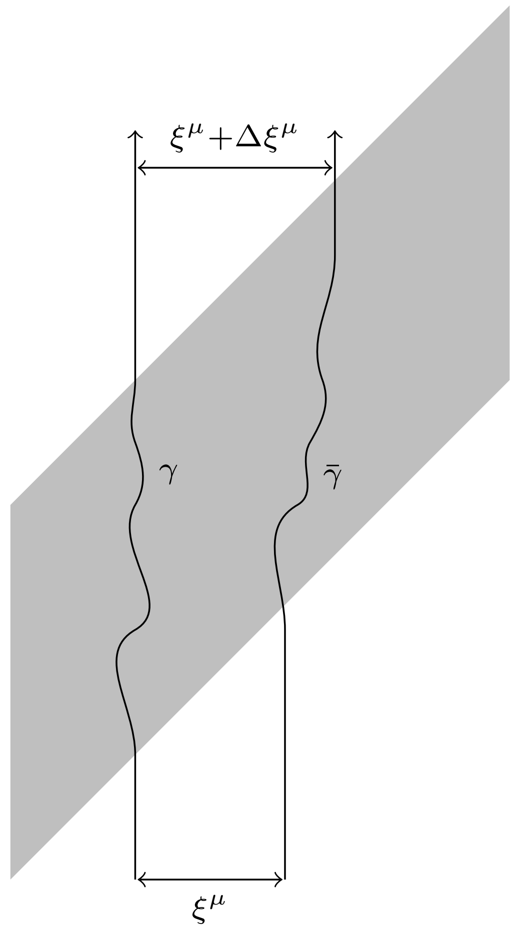

Figure 1.4: A depiction of the gravitational wave memory effect, a change $\Delta\xi^\mu$ in the separation two closely-separated observers with worldlines γ and $\bar{\gamma}$. This change in separation is due to the gravitational wave burst, represented in this diagram by the shaded region.

theory for a bound system (see, for example, [164]), both bound and unbound systems in *nonlinear* general relativity generate the memory effect [53]. In this context, there is an additional effect (known as nonlinear or sometimes null memory [32]) arising from perturbations generated by the effective stress-energy of the gravitational waves itself, or of other massless fields that can propagate to infinity [186, 164, 167]. In particular, the memory effect is expected to occur in neutrino emission and kicks during core-collapse supernovae (for example, [169, 45]), emission of matter during certain gamma-ray bursts (for example, [147, 144]), and compact binary mergers (for example, [64, 132, 65]).

Early descriptions of the types of experiments needed to detect the gravitational wave memory effect were laid out by Braginsky and Grishchuk [41] and Braginsky and Thorne [40]. Currently, searches for these bursts have been carried out using pulsar timing arrays, which have provided constraints on their frequency of occurrence [180, 15]. Here, the effect of a burst of gravitational waves with memory will be correlated discontinuities in the measured frequency of the pulsars, since the bursts that produce the memory effect are much shorter than the period between measurements [172]. Moreover, by stacking the signals of many compact binary mergers, it may also be possible to detect the gravitational wave memory with the LIGO and Virgo detectors [112]. For LISA, the gravitational wave memory effect may be able to be measured from a single detection [66].

One may ask about the relevance of angular momentum to the gravitational wave memory effect. The idea is given by the following rough argument: suppose that two observers, which we label "1" and "2", measure the angular momentum of a distant object:

$$\vec{L}_1 = \vec{r}_1 \times \vec{p}, \qquad \vec{L}_2 = \vec{r}_2 \times \vec{p}, \tag{1.64}$$

where $\vec{p}$ is the momentum of the distant object and $\vec{r}_1$ and $\vec{r}_2$ are the observers' locations with respect to that object. Suppose, however, that unbeknownst to these two observers, a gravitational with memory passes by their locations, and so their relative separation would change by an amount $\Delta\vec{\xi}$. This change can be determined by measuring the angular momentum of the distant source once again. Since

$$\vec{L}_2 - \vec{L}_1 = (\vec{r}_2 - \vec{r}_1) \times \vec{p}, \tag{1.65}$$

we have that, denoting this second set of measurements by $\vec{L}'_1$ and $\vec{L}'_2$,

$$\vec{L}'_2 - \vec{L}'_1 = \vec{L}_2 - \vec{L}_1 + \Delta\vec{\xi} \times \vec{p}, \tag{1.66}$$

assuming the momentum of the distant object remains unchanged. Of course, this discussion merely *suggests* a connection between the memory effect and angular momentum—the use of angular momentum to describe the memory effect and its generalizations is given in chapter 4.

1.4 Summary of this Thesis

We now provide a summary of the content of this thesis. In chapters 2 and 3, we consider the Carter constant mentioned above in section 1.2.2, a constant of motion for point particles in the Kerr spacetime that generalizes the notion of "total angular momentum". The goal of these chapters is to understand the behavior of point particles in such spacetimes, as discussed above in section 1.3.3.1. In particular, the question that we hope to resolve is whether there exist conserved currents for field theories in the Kerr spacetime that are associated with the Carter constant of a point particle, in the same way that there are conserved currents associated with the constants of motion that arise from isometries (energy and axial angular momentum).

In chapter 2, containing work done with Éanna Flanagan in [81], we show that, for an arbitrary theory which is characterized only by its stress-energy tensor, there exists no such conserved current. In particular, we show that there is no quantity that is

- constructed from the stress-energy tensor,
- conserved for arbitrary theories, only relying on the conservation of stress-energy, and
- reduces to the Carter constant in the case where the stress-energy tensor represents that of a point particle.

Our proof relies on the construction of a numerical counterexample, using a theory that contains a point particle that can decay into two point particles, which later fuse back together. If such a general conserved quantity outlined above were to exist, the Carter constant of the initial and final particle would be the same, and yet one can easily construct an example in which they are not.

The work presented in chapter 2, however, does not forbid the existence of conserved currents that are associated with the Carter constant for particular theories—it merely forbids currents that are constructed from the stress-energy tensor. For specific field theories it may be possible to find such currents in terms of the fields. In particular, there exists a conserved current for complex scalar fields, first discovered by Carter [49]. In the limit where solutions to the scalar field equations represent a null fluid of scalar quanta (the limit of *geometric optics*), the current is related to the Carter constants of the individual quanta. In chapter 3, containing work done with Éanna Flanagan in [83], we construct a variety of conserved currents for linearized gravity on the Kerr background, all of which are associated, in the geometric optics limit, with the Carter constants of gravitons. What we do not show, however, is that the fluxes of these currents through infinity and the horizon are related to the change in the Carter constant of a point particle. However, we identify which of these currents possess finite fluxes, and compute these fluxes in the hope that they may one day be useful.

Chapter 4 switches topics entirely, focusing on generalizations of the gravitational wave memory effect summarized above in section 1.3.3.2. This chapter contains work with Éanna Flanagan, Abraham Harte, and David Nichols in [67, 70]. The motivation for these generalization is in the discovery of associations between the gravitational wave memory effect, asymptotic symmetries of gravity, and soft theorems in quantum field theory (see the review [156] and references therein). Analogous effects have also been shown to occur in electromagnetism and quantum chromodynamics, and consequences of the latter may be seen in electron-ion colliders [26]. Even in the case of general relativity, new types of memory effects have been proposed, such as permanent changes in the relative velocity of freely falling observers after a burst of gravitational waves (see, for example, [68]). Our addition to this field has been in defining a broad collection of generalizations to the gravitational wave memory, each sharing one key aspect with this effect: they can be (in principle) measured by a system of observers over a given duration, and they "persist" after a burst of gravitational waves. As such, we call the generalizations that we have proposed *persistent observables*.

In this chapter we consider three observables which we defined in [67]: a generalization of the memory effect that allows for non-comoving (and even accelerating) observers, a geometric object known as a *holonomy* that is related to angular momentum, and finally a collection of quantities that can be measured using a spinning test mass. In the bulk of this chapter, we provide explicit methods for calculating these observables in arbitrary spacetimes. We also include explicit calculations in *nonlinear* plane wave spacetimes, which are exact solutions to the Einstein equations that represent gravitational waves.

The final topic covered in this thesis, in chapter 5, is that of angular momentum in Einstein-Maxwell theory, which contains work with Béatrice Bonga and Kartik Prabhu in [38]. The general idea is that there exists an oddity in the theory of angular momentum in electromagnetism (even in flat space): the flux of angular momentum, as defined using the stress-energy tensor reviewed above in section 1.1.3, depends not only on the radiative ($1/r$) fields, but also the Coulombic ($1/r^2$) fields. This is in contrast to energy and linear momentum, which only depend on the radiative fields. In this chapter, we first show that there are other definitions of conserved currents, the Noether and symplectic currents mentioned in section 1.1.3, whose fluxes for energy, linear momentum, *and* angular momentum all solely depend on the radiative fields. Which notion of these conserved currents is "correct" is ill-defined; instead, they are all useful for solving different problems.

A case where the Noether current (and not the current defined by the stress-energy tensor) is relevant is in considering notions of mass and angular momentum in general relativity coupled to electromagnetism, or *Einstein-Maxwell theory*. As mentioned above in section 1.1.4, mass and angular momentum are charges in general relativity: integrals over spheres at null infinity that are related to symmetries, and are constant in non-radiating spacetimes. Moreover, in vacuum general relativity, their fluxes are only dependent on radiative degrees of freedom in the gravitational field. In Einstein-Maxwell theory, on the other hand, using the "usual" notions of energy and angular momentum from vacuum general relativity, one finds that their fluxes depend on the conserved currents that are constructed from the stress-energy tensor. Therefore, the flux of angular momentum in Einstein-Maxwell theory is *not* dependent on purely radiative fields, as it was in vacuum general relativity! This suggests that this notion of angular momentum is somehow incorrect. As an alternative approach, we instead use the Wald-Zoupas procedure described above in section 1.1.4 to determine the charges associated with the asymptotic symmetries of Einstein-Maxwell theory. We find that these charges are modified in Einstein-Maxwell theory for the case of asymptotic rotations and boosts, and this new notion of angular momentum in Einstein-Maxwell theory cannot simply be determined from the gravitational fields, since it requires knowledge of the electromagnetic fields as well. The fluxes of these Wald-Zoupas charges depend not on the stress-energy current, but the Noether current, and are therefore purely radiative.

Finally, we list the conventions used throughout this thesis: lowercase Latin indices from the beginning of the alphabet (a, b, etc.) refer to abstract indices, whereas lowercase Greek letters refer to coordinate indices. Tensors, when appearing without indices, are in bold. Unless otherwise specified, we follow the conventions of Wald [176] for the metric g_{ab}, Riemann tensor $R^a{}_{bcd}$, and differential forms.

Chapter 2

The Carter Constant for General Theories

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As mentioned in the introduction, freely falling point particles in the Kerr spacetime are known to possess a constant of motion, the *Carter constant* [48], which is not associated with either of the two isometries (stationarity and axisymmetry) of the spacetime. The existence of four constants of motion in the Kerr spacetime—energy, the z component of angular momentum (axial angular momentum), the Carter constant, and the mass of the particle—allow for the solution of the geodesic equation in terms of first-order differential equations. As such, the Carter constant is very important for understanding the behavior of point particles in the Kerr spacetime.

Consider now point particles coupled to fields in the Kerr spacetime. Do the conserved quantities discussed above generalize to this setting? The energy and axial angular momentum do generalize, since they are associated with isometries. Specifically, given the infinitesimal symmetry generator ξ^a, one can always construct a conserved current of the form $T^{ab}\xi_b$, where T^{ab} is the stress-energy tensor of the system (particle and/or field). Is there such a generalized conservation law that is associated with the Carter constant? There are a few cases where such generalized conservation laws exist:

- There exists a generalization of the Carter constant for charged particles in rotating charged black hole spacetimes [48], as well as spinning test particles, to linear order in the spin [141][1].

[1] The motion of a spinning point particle in the Kerr spacetime is thus integrable to linear order in spin [94]. This does not contradict the fact that chaotic behavior is seen in numerical studies of spinning point particle dynamics [110], since that behavior is due to effects that are higher order in spin.

- For a free scalar field on the Kerr background, Carter showed [49] that one can construct a conserved current using the same Killing tensor K_{ab} that enters into the definition of the Carter constant [equation (1.43)]. While we will review this conserved current in more detail in the next chapter, we note that this conserved current provides a generalization of the Carter constant in the following sense: a solution to the scalar field equation of the form $\Phi \propto \exp(-i\vartheta/\epsilon)$, in the geometric optics limit $\epsilon \to 0$, can be interpreted as a null fluid of scalar quanta. Integrating the conserved current of [49] over a hypersurface, one finds in this limit that this integral reduces to a sum of the Carter constants of scalar quanta. This is valid for both massive and massless fields. Moreover, a similar conserved current was found in [47] for the case of spin-1/2 fields, and as we will discuss in chapter 3, there are additional examples in the case of electromagnetism and linearized gravity on the Kerr background.

- Ashtekar and Kesavan have shown that in spacetimes which settle down at late times to a Kerr black hole, the Killing tensor at future null infinity can be expressed as a linear combination of products of asymptotic symmetry vector fields (BMS generators), allowing them to compute a charge associated with any cut and derive an asymptotic conservation law [17].

In this chapter, we consider a different possible type of generalized conservation law, namely the existence of a quantity that is conserved under local interactions between particles that obey stress-energy conservation. Specifically, suppose we are given a conserved stress-energy tensor T_{ab} in the Kerr spacetime with compact spatial support. Does there exist a quantity $\mathcal{K}_\Sigma$ which can be computed from T_{ab} and its derivatives on any Cauchy hypersurface Σ, which has the properties that (i) $\mathcal{K}_\Sigma$ is independent of Σ, and (ii) $\mathcal{K}_\Sigma$ reduces to the Carter constant when T_{ab} represents the stress-energy of a point particle?

We show that no such quantity $\mathcal{K}_\Sigma$ exists. In particular, we construct a numerical counterexample in a very specific theory: one that allows a single freely falling point particle to decay into two particles, which travel along geodesics before they fuse back together when their geodesics intersect. As we will show in sections 2.2 and 2.3 below, the Carter constant of the final particle can differ from that of the initial one, disproving the existence of a general conservation law.

The remainder of this chapter is organized as follows. In section 2.1, we first review the theory of geodesics in the Kerr spacetime, including how the Carter constant arises in the equations of motion. We then show in section 2.2 how the existence of the quantity $\mathcal{K}_\Sigma$, which generalizes the Carter constant, implies an easily testable property of intersecting geodesics in the Kerr spacetime. This condition is tested in section 2.3, and it is shown to be false. We conclude with some brief discussion in section 2.4.

2.1 Geodesics in the Kerr Spacetime

In this section, we briefly review the equations satisfied by geodesics in the Kerr spacetime, roughly following [125]. In the course of this review, we will also show how the Carter constant appears, allowing for the geodesic equations to be written as a collection of first-order ordinary differential equations in terms of four constants of motion.

The geodesic equation in the Kerr spacetime can be written down starting with the metric in *Boyer-Lindquist* coordinates:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) + (r^2+a^2)\sin^2\theta\mathrm{d}\phi^2 + \frac{r^2+a^2-\Delta}{\Sigma}\left(a\sin^2\theta\mathrm{d}\phi - \mathrm{d}t\right)^2, \tag{2.1}$$

where

$$\Sigma \equiv r^2 + a^2\cos^2\theta, \qquad \Delta \equiv r^2 - 2Mr + a^2, \tag{2.2}$$

and where M is the mass of the black hole and Ma its angular momentum. Due to the t- and ϕ-independence of the metric, there are two Killing vectors in this spacetime, $t^a \equiv (\partial_t)^a$ and $\phi^a \equiv (\partial_\phi)^a$. As such, there are two constants of motion, the energy and (axial) angular momentum per unit mass,

$$\tilde{E} \equiv -t_a\dot{\gamma}^a, \qquad \tilde{L}_z \equiv \phi_a\dot{\gamma}^a, \tag{2.3}$$

where $\dot{\gamma}^a$ is the tangent to the geodesic, normalized such that

$$q \equiv \dot{\gamma}^a\dot{\gamma}_a \tag{2.4}$$

is ± 1 or 0. In this chapter, it is often useful to consider conserved quantities by using the four-velocity $\dot{\gamma}^a$, instead of the momentum p^a. The quantities defined using the four-velocity we will denote with tildes, while those defined in terms of the momentum will not have tildes: for example, $E \equiv -t_a p^a = m\tilde{E}$, where m is the mass of the particle.

One can invert these definitions (2.3) of $\tilde{E}$ and $\tilde{L}_z$ to solve for $\mathrm{d}t/\mathrm{d}\tau$ and $\mathrm{d}\phi/\mathrm{d}\tau$:

$$\Sigma\frac{\mathrm{d}t}{\mathrm{d}\tau} = aD(\theta) + \frac{r^2+a^2}{\Delta}P(r), \tag{2.5}$$

$$\Sigma\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{D(\theta)}{\sin^2\theta} + \frac{a}{\Delta}P(r), \tag{2.6}$$

where

$$D(\theta) \equiv \tilde{L}_z - a\tilde{E}\sin^2\theta, \tag{2.7}$$

$$P(r) \equiv \tilde{E}(r^2+a^2) - a\tilde{L}_z. \tag{2.8}$$

Next, we must find $\mathrm{d}r/\mathrm{d}\tau$ and $\mathrm{d}\theta/\mathrm{d}\tau$. To do so, note that from equations (2.5) and (2.6),

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} - a\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{1}{\Delta}P(r), \tag{2.9}$$

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - (r^2+a^2)\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = \frac{1}{\Sigma}\left[-\frac{D(\theta)^2}{\sin^2\theta} + \frac{r^2+a^2}{\Delta^2}P(r)^2\right], \tag{2.10}$$

and so one finds from the normalization (2.4) of $\dot{\gamma}^a$ that

$$q\Sigma = \Sigma^2\left[\frac{1}{\Delta}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2\right] + \frac{D(\theta)^2}{\sin^2\theta} - \frac{1}{\Delta}P(r)^2. \tag{2.11}$$

At this point, define *Mino time* λ by [119]

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \Sigma. \tag{2.12}$$

Using Mino time instead of proper time, we find that we can rearrange equation (2.11) to yield

$$qr^2 - \frac{1}{\Delta}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{1}{\Delta}P(r)^2 = \left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + qa^2\cos^2\theta + \frac{D(\theta)^2}{\sin^2\theta}. \tag{2.13}$$

In the Kerr spacetime, there exists a Killing tensor K_{ab}, satisfying the Killing tensor equation (1.18), that is given by

$$K_{ab} = r^2 g_{ab} + \frac{1}{\Delta}l_{(a}n_{b)}, \tag{2.14}$$

where

$$l^a = (r^2+a^2)t^a + a\phi^a + \Delta(\partial_r)^a, \qquad n^a = (r^2+a^2)t^a + a\phi^a - \Delta(\partial_r)^a. \tag{2.15}$$

It is relatively easy to show using equation (2.1) that the right-hand side of equation (2.13) is then equal to the Carter constant $\tilde{K}$ per unit mass squared:

$$\tilde{K} \equiv K_{ab}\dot{\gamma}^a\dot{\gamma}^b. \tag{2.16}$$

Since K_{ab} is a Killing tensor, $\tilde{K}$ is a constant, and so we find that the geodesic equations for r and θ (in Mino time) take the following simple form:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \Delta(qr^2 - \tilde{K}) + P(r)^2, \tag{2.17}$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 = \tilde{K} - qa^2\cos^2\theta - \frac{D(\theta)^2}{\sin^2\theta}. \tag{2.18}$$

Occasionally, it is useful to instead consider

$$\tilde{Q} \equiv \tilde{K} - (a\tilde{E} - \tilde{L}_z)^2, \tag{2.19}$$

which [by equations (2.18) and (2.7)] vanishes for equatorial orbits.

To provide an interpretation of the Carter constant, it is useful to consider the case of Schwarzschild, where one can set $a = 0$ in equation (2.18), yielding (in terms of $\mathrm{d}\theta/\mathrm{d}\tau$)

$$m^2r^4\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 = K - \frac{L_z^2}{\sin^2\theta}, \tag{2.20}$$

where we have re-inserted the factors of the mass m of the particle. Now, due to spherical symmetry, Schwarzschild has two Killing vectors not present in Kerr; this yields two additional angular momenta L_x and L_y given by

$$L_x = -mr^2\sin\phi\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \cot\theta\cos\phi L_z, \tag{2.21}$$

$$L_y = mr^2\cos\phi\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \cot\theta\sin\phi L_z. \tag{2.22}$$

As such, we have that

$$L^2 = L_x^2 + L_y^2 + L_z^2 = m^2r^4\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \frac{L_z^2}{\sin^2\theta} = K. \tag{2.23}$$

As mentioned in the introduction, the Carter constant is therefore a generalization of total angular momentum.

2.2 The Carter Constant and the Decay of Point Particles

Consider a point particle which decays at some point x into two particles, conserving four-momentum. Its two decay products then freely fall until their geodesics meet at another point

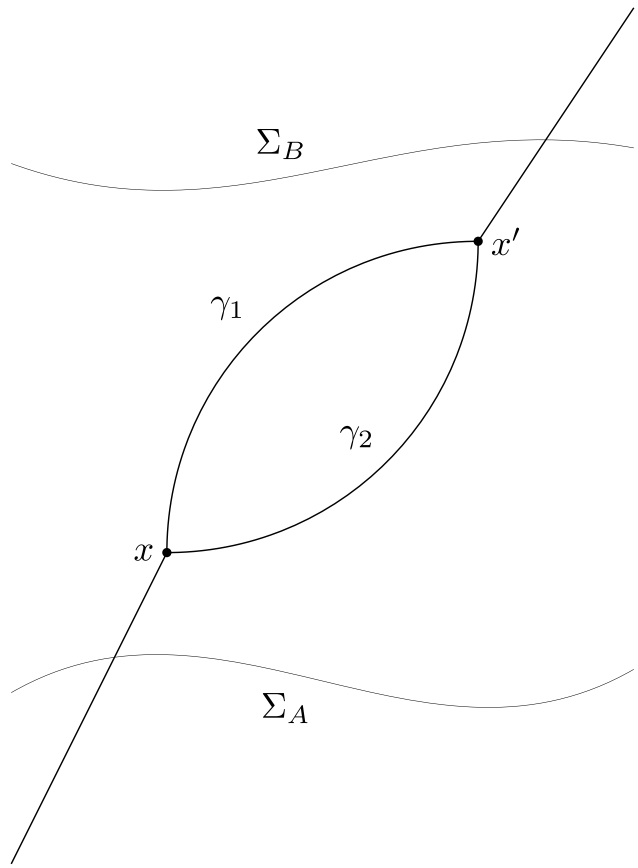

Figure 2.1: An illustration of a process obeying stress-energy conservation that demonstrates the non-existence of any "generalized Carter constant" $\mathcal{K}_\Sigma$ that is constructed from the stress-energy tensor. An initial point particle decays at x to yield two decay products which follow geodesics γ_1 and γ_2 until the point x', at which they collide and fuse to form a final point particle. The Carter constant of the initial particle can be evaluated where it crosses the hypersurface Σ_A, and the final particle's Carter constant similarly evaluated where it crosses Σ_B. These two Carter constants would coincide if a generalized Carter constant $\mathcal{K}_\Sigma$ existed. We give examples where the initial and final Carter constants differ in section 2.3.

x', at which point they fuse to form a single particle again. This setup is shown in figure 2.1. Throughout this process, all particles are assumed to follow geodesics. We now show that, if a conserved quantity $\mathcal{K}_\Sigma$ of the sort introduced at the beginning of this chapter were to exist, then there would be a rather stringent constraint on the behavior of the geodesics followed by the decay products.

As such, let p^a be the four-momentum of the initial particle at x, and let $p^{a'}$ be the four-momentum of the final particle at x'. By stress-energy conservation, one has that

$$p^a = p_1^a + p_2^a, \qquad p^{a'} = p_1^{a'} + p_2^{a'}, \tag{2.24}$$

where p_1^a and p_2^a are the four-momenta of the two decay products at x, and $p_1^{a'}$ and $p_2^{a'}$ their four-momenta at x'. By contracting equation (2.24) with the Killing vector t^a, we find the following relationship between the initial and final energies:

$$E = E_1 + E_2 = E', \tag{2.25}$$

where E is the energy of the initial particle, E_1 and E_2 are the the energies of the two decay products, and E' is the energy of the final particle. This equation holds because E_1 and E_2 are both linear in momentum and are conserved along the trajectories of the two decay products. A similar relationship holds for the axial angular momenta.

For the Carter constant, on the other hand, we have that the Carter constant of the initial particle is

$$K = K_{ab}(p_1^a + p_2^a)(p_1^b + p_2^b) = K_1 + K_2 + 2K_{ab}p_1^a p_2^b. \tag{2.26}$$

Similarly, the Carter constant of the final particle is

$$K' = K_1 + K_2 + 2K_{a'b'}p_1^{a'} p_2^{b'}, \tag{2.27}$$

and so $K = K'$ if and only if

$$K_{ab}p_1^a p_2^b = K_{a'b'}p_1^{a'} p_2^{b'}. \tag{2.28}$$

and this equality does not obviously hold. However, if the generalized Carter constant $\mathcal{K}_\Sigma$ were to exist, then K and K' would necessarily be equal. This is because $\mathcal{K}_\Sigma$ reduces to the Carter constant for a point particle by assumption, and evaluating $\mathcal{K}_\Sigma$ at the hypersurfaces Σ_A (before the decay) and Σ_B (after the fusion) gives

$$\mathcal{K}_{\Sigma_A} = K, \qquad \mathcal{K}_{\Sigma_B} = K'. \tag{2.29}$$

Since $\mathcal{K}_\Sigma$ is, by definition, independent of hypersurface, we obtain $K = K'$.

To show that there exists no conserved quantity $\mathcal{K}_\Sigma$, it is therefore sufficient to find a pair of timelike geodesics, γ_1 and γ_2, intersecting at two points x and x' and violating equation (2.28). In the next section, we present the details of finding geodesics that intersect in this way.

2.3 A Numerical Counterexample

In the previous section, we have reduced answering the posed question in the beginning of this chapter to finding two intersecting geodesics in Kerr that violate equation (2.28).

It is relatively straightforward to find geodesics which intersect at two distinct points, in a few specific cases. Spherical ($\mathrm{d}r/\mathrm{d}\tau = 0$) or equatorial ($\mathrm{d}\theta/\mathrm{d}\tau = 0$, $\theta = \pi/2$) geodesics that intersect at two points are relatively straightforward to find, due to the reduced dimensionality of the problem. Another class of geodesics where intersections are easy to find are the following: if x is a point on the equatorial plane, and the components of the four-velocities of two geodesics that pass through x are equal except for $\mathrm{d}\theta_1/\mathrm{d}\tau = -\mathrm{d}\theta_2/\mathrm{d}\tau$, then if either of these geodesics crosses the equatorial plane at another point x', then the two geodesics will intersect at x'. We call such pairs of geodesics *reflected geodesics.*

However, these geodesics all have the property that the initial and final Carter constants K and K' must coincide. This is because, in the equatorial and reflected case, the four-velocities of the initial and final particles satisfy

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{\mathrm{d}\theta'}{\mathrm{d}\tau} = 0, \qquad \theta = \theta' = \pi/2, \tag{2.30}$$

and similarly for the spherical case,

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r'}{\mathrm{d}\tau} = 0, \qquad r = r'. \tag{2.31}$$

Using equations (2.17) and (2.18), it is then clear in both of these cases that $K = K'$.

As such, these simple examples of intersecting geodesics are not sufficient. Instead, we turn to the following approach, inspired by the technique of *shooting* in solving boundary value problems (see, for example, [137]). Start with a pair of geodesics of the above types that intersect at two points, x and x'. Then, adjust the four-momenta of each of these two geodesics at x by a small amount. The two geodesics will then no longer intersect at x', but their final points should be close. We denote the coordinate difference between the final point of γ_1 and γ_2 by δx^μ. Fix one geodesic, for example γ_1, and consider small perturbations δp_2^a to p_2^a. The coordinate separation between the final locations of γ_2 before and after such a perturbation, to linear order, is given by

$$\delta x_2^\mu \simeq H^\mu{}_\nu \delta p_2^\nu, \tag{2.32}$$

for some matrix $H^\mu{}_\nu$. By considering a number of perturbations δp_2^ν, one can numerically determine a fairly accurate value for $H^\mu{}_\nu$. One can then compute a δp_2^μ that brings the endpoint of γ_2 close to that of γ_1 via

$$\delta p_2^\mu = -(H^{-1})^\mu{}_\nu \delta x^\nu. \tag{2.33}$$

This process can be iterated to yield a δx^ν that is small enough to be considered an intersection (we considered a $|\delta \boldsymbol{x}| \lesssim 10^{-12} M$, where M was the mass of the black hole, to be sufficiently close).

In the case of equatorial or spherical initial geodesics, we found that this process led to $H^\mu{}_\nu$'s that were nearly singular, and so this method was not useful. However, for the reflected geodesics discussed above, we managed to successfully perturb these geodesics such that x' was sufficiently far off of the equatorial plane that we could definitively say that these geodesics provided a numerical counterexample. The full numerical counterexample is given as follows: consider a Kerr spacetime, representing a spinning black hole of mass M and spin parameter $a = 0.84M$. The initial and final points are given by

$$t = 0, \qquad r \simeq 3.508M, \qquad \theta = \pi/2, \qquad \phi = 0, \tag{2.34a}$$

$$t' \simeq 51.52M, \qquad r' \simeq 3.524M, \qquad \theta' \simeq 1.557, \qquad \phi' \simeq 7.821, \tag{2.34b}$$

where (for brevity) we only keep four significant figures (for far more significant figures than probably relevant, see appendix A of [81]). The constants of motion for each of these geodesics are given by

$$\tilde{E}_1 \simeq 0.9212, \qquad \tilde{L}_{z,1} \simeq 1.424M, \qquad \tilde{Q}_1 \simeq 6.692M^2, \tag{2.35a}$$

$$\tilde{E}_2 \simeq 0.9284, \qquad \tilde{L}_{z,2} \simeq 1.177M, \qquad \tilde{Q}_2 \simeq 8.051M^2. \tag{2.35b}$$

As this description of initial data does not give the signs of $\mathrm{d}\theta/\mathrm{d}\tau$ or $\mathrm{d}r/\mathrm{d}\tau$, note that $\mathrm{d}\theta_1/\mathrm{d}\tau > 0$ and $\mathrm{d}\theta_2/\mathrm{d}\tau < 0$, and similarly that $\mathrm{d}r_1/\mathrm{d}\tau > 0$ and $\mathrm{d}r_2/\mathrm{d}\tau > 0$. Over the course of integrating the first-order geodesic equations, we ensured that these constants of motion remained constant up to a relative error of 10^{-12}. This integration was performed over a range of proper times from $\tau_1 = \tau_2 = 0$ to $\tau_1' \simeq 26.80M$ and $\tau_2' \simeq 26.73M$.

With these initial and final data, the Carter constants of the initial and final particles were given by $K \simeq 1.159m^2M^2$ and $K' \simeq 1.161m^2M^2$, where m is the (assumed equal) mass of the decay products. A dimensionless measure of their difference is given by

$$\frac{2(K - K')}{K + K'} \simeq -1.422 \times 10^{-3}. \tag{2.36}$$

Since the constants of motion of these geodesics were constant within a relative error of 10^{-12}, this shows that the difference between K and K' is not due to numerical error. Similarly, performing these integrations with a range of relative error tolerances between 10^{-13} and 10^{-15}, the quantity in equation (2.36) had a standard deviation[2] of the order 10^{-12}.

The above result confirms that there is no such quantity $\mathcal{K}_\Sigma$ which reduces to the Carter constant. One may ask about another constant of motion, namely the square of the mass, which is associated with the trivial Killing tensor, the metric g_{ab} [as $\nabla_a g_{bc} = 0$, it clearly obeys equation (1.18)]. We now show that there is also no quantity $\mathcal{M}_\Sigma$ that obeys the same requirements as $\mathcal{K}_\Sigma$ but with "Carter constant" replaced by "squared mass". This can be seen by comparing the masses m_{tot} and m'_{tot} before and after the decay; in this numerical counterexample,

$$\frac{2(m_{\mathrm{tot}}^2 - m_{\mathrm{tot}}'^2)}{m_{\mathrm{tot}}^2 + m_{\mathrm{tot}}'^2} \simeq 3.279 \times 10^{-3}. \tag{2.37}$$

That is, in the process of decay and fusion, this particle lost mass.

As a final test, we considered this same process in the Schwarzschild spacetime, where we expect the Carter constants of the initial and final particles to be the same, as they are both equal to

[2] If I have one question for my past self, it is "why did you not perform a real convergence test here. . . ?"

squared angular momentum:

$$K = L_x^2 + L_y^2 + L_z^2 = L_x'^2 + L_y'^2 + L_z'^2 = K', \tag{2.38}$$

where the second equality holds since each of L_x, L_y, and L_z are related to Killing vectors. In the Schwarzschild spacetime, we needed to use different initial data: the initial point was the same, but the final point was given by

$$t' \simeq 33.75M, \qquad r' \simeq 8.258M, \qquad \theta' \simeq 1.828, \qquad \phi' \simeq 3.125. \tag{2.39}$$

The constants of motion of the two decay products in this case were given by

$$\tilde{E}_1 \simeq .9955, \qquad \tilde{L}_{z,1} \simeq 0.2373M, \qquad \tilde{Q}_1 \simeq 14.46M^2, \tag{2.40a}$$

$$\tilde{E}_2 \simeq 1.059, \qquad \tilde{L}_{z,2} \simeq -0.2729M, \qquad \tilde{Q}_2 \simeq 19.12M^2, \tag{2.40b}$$

and the final proper times were $\tau_1' \simeq 20.88M$ and $\tau_2' \simeq 18.78M$. In this case, we found that

$$\frac{2(K-K')}{K+K'} \simeq -7.940 \times 10^{-14}, \qquad \frac{2(m_{\text{tot}}^2 - m_{\text{tot}}'^2)}{m_{\text{tot}}^2 + m_{\text{tot}}'^2} \simeq 0.6175. \tag{2.41}$$

The first of these two expressions is below our relative error tolerance of 10^{-12}, and so is consistent with K and K' being equal. The second expression shows that the "non-conservation of mass" is still present in Schwarzschild.

2.4 Discussion

In this chapter, we have shown that there is no conserved quantity $\mathcal{K}_\Sigma$ computed from a stress-energy tensor on a Cauchy hypersurface Σ that generalizes the Carter constant for point particles. In particular, this means that there is no conserved current (like $T^{ab}\xi_b$ defined for Killing vectors) that is constructed from the Killing tensor K_{ab} and an arbitrary stress-energy tensor, such that the integral of the current over a hypersurface reduces to the Carter constant when the stress-energy tensor is that of a point particle. This statement about conserved currents is a special case of our result. If there were a conserved current j_K^a associated with the Carter constant for an arbitrary stress-energy tensor in the Kerr spacetime, then $\mathcal{K}_\Sigma$ could be given by

$$\mathcal{K}_\Sigma = \int_\Sigma j_K^a \mathrm{d}\Sigma_a. \tag{2.42}$$

However, we have also eliminated nonlinear conserved quantities that cannot be written in this form.

In this chapter, we consider only conserved quantities constructed from stress-energy tensors. In the next chapter, we will consider a more general framework, and discuss conserved currents constructed for linearized gravity in the Kerr spacetime.

Chapter 3

The Carter Constant for Linearized Gravity

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The results of the previous chapter imply that there do not exist conserved quantities generalizing the Carter constant for point particles that are associated with arbitrary stress-energy tensors on a Kerr background. However, these results do not eliminate the possibility of conserved currents that are not constructed from a stress-energy tensor. In particular, there are already known conserved currents for scalar, spin-1/2, and electromagnetic test fields:

- For a sourceless complex scalar field Φ, the conserved charge is the Klein-Gordon inner product of Φ with ${}_0\mathcal{D}\Phi$ [49]:

$$ {}_0K \equiv \frac{1}{2i} \int_\Sigma \mathrm{d}^3\Sigma^a \left[({}_0\mathcal{D}\Phi)\nabla_a\overline{\Phi} - \overline{\Phi}\nabla_a {}_0\mathcal{D}\Phi\right], \tag{3.1} $$

  where Σ is any spacelike hypersurface, bars denote complex conjugation, and the differential operator ${}_0\mathcal{D}$ is defined by

$$ {}_0\mathcal{D}\Phi \equiv \nabla_a(K^{ab}\nabla_b\Phi), \tag{3.2} $$

  where K_{ab} is the Killing tensor in Kerr in equation (1.43), defined in more detail in equation (3.8) below. The operator ${}_0\mathcal{D}$ commutes with the d'Alembertian, and so maps the space of solutions into itself. The charge ${}_0K$ is associated with the Carter constant in the following sense: for a solution of the form $\Phi \propto e^{-i\vartheta/\epsilon}$, which represents a collection of scalar quanta with Carter constants $\{K_\alpha\}$, the charge is given by (in the geometric optics limit $\epsilon \to 0$)

$$_0K = \frac{1}{\hbar}\sum_\alpha K_\alpha. \tag{3.3}$$

That is, the charge is proportional to the sum of the Carter constants of each scalar quantum. In the case of real scalar fields, the charge vanishes in the geometric optics limit.

- A similar result holds for any spin-1/2 field ψ satisfying the Dirac equation [47]. An operator $_{1/2}\mathcal{D}$, which is defined in terms of the Killing-Yano tensor f_{ab} and commutes with the Dirac operator, is given by

$$_{1/2}\mathcal{D} = i\gamma_5\gamma^a\left(f_a{}^b\nabla_b - \frac{1}{6}\gamma^b\gamma^c\nabla_c f_{ab}\right), \tag{3.4}$$

  where γ^a is the usual gamma matrix and, in terms of the Levi-Civita tensor ϵ_{abcd}, $\gamma_5 \equiv i\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d$. The charge which generalizes the charge in equation (3.1) is proportional to the following integral over a spacelike hypersurface Σ:

$$_{1/2}K \propto \int_\Sigma \mathrm{d}^3\Sigma_a\, \overline{(_{1/2}\mathcal{D}\psi)}\gamma^a\, {}_{1/2}\mathcal{D}\psi. \tag{3.5}$$

  As in the scalar field case, this charge is proportional to the sum of the Carter constants of the individual quanta in the geometric optics limit. This construction works for massive as well as massless spin-1/2 particles, and even charged spin-1/2 particles in the case of the Kerr-Newman spacetime [47].

- For electromagnetic fields, there are several conserved charges which satisfy the requirement of reducing, in the geometric optics limit, to a sum of (some power) of the Carter constants of the photons; some examples are given by [14], which we have considered in [82] (along with additional examples).

It would be interesting to find similar conserved currents in the case of linearized gravity. One motivation for considering such conserved currents was mentioned in the introduction, in the context of the extreme mass-ratio inspiral problem (see section 1.3.3.1), since such a conserved current may allow for the computation of evolution of the Carter constant due to the emission of gravitational waves.

Table 3.1: Summary of the properties of the conserved currents considered in this chapter. For convenience, we give the equation numbers (within section 3.3.2) in which these currents are defined. We then give the limit of the corresponding charges in geometric optics, where K is the Carter constant of a graviton (see section 3.4 for the definitions of the polarization coefficients e_R and e_L, as well as the justification of the factors of $\hbar$). The next column indicates whether the fluxes of these currents through future and past null infinity ($\mathscr{I}^{\pm}$) and the future and past horizons ($H^{\pm}$) are finite. We finally indicate which of these currents are local functionals of the metric perturbation.

	Definition	Geometric optics limit	Finite fluxes?				
Current	(equation)	of charge (per graviton)	$\mathscr{I}^+$	$\mathscr{I}^-$	H^+	H^-	Local?
${}_{2\mathcal{C}}j^a[\mathring{\delta}\boldsymbol{g}]$	(3.144)	$K^4(\lvert e_R\rvert^2 - \lvert e_L\rvert^2)/\hbar^7$	×	✓	✓	✓	✓
${}_{-2\mathcal{C}}j^a[\mathring{\delta}\boldsymbol{g}]$			✓	×	✓	✓	✓
${}_{2\mathring{\mathcal{C}}}j^a[\mathring{\delta}\boldsymbol{g}]$	(3.145)	$K^4(\lvert e_R\rvert^2 - \lvert e_L\rvert^2)/\hbar^7$ [a]	✓	✓	✓	✓	×
${}_{2\mathcal{D}}j^a[\mathring{\delta}\boldsymbol{g}]$	(3.146)	$K(\lvert e_R\rvert^2 - \lvert e_L\rvert^2)/\hbar$	×	✓	✓	✓	×
${}_{-2\mathcal{D}}j^a[\mathring{\delta}\boldsymbol{g}]$			✓	×	✓	✓	×
${}_{2\mathring{\mathcal{D}}}j^a[\mathring{\delta}\boldsymbol{g}]$	(3.147)	$K(\lvert e_R\rvert^2 - \lvert e_L\rvert^2)/\hbar$ [a]	✓	✓	✓	✓	×
${}_{2\Omega}j^a[\mathring{\delta}\boldsymbol{g}]$	(3.148)	$K^4(\lvert e_R\rvert^2 - \lvert e_L\rvert^2)/\hbar^7$	✓	✓	✓	✓	✓
${}_{-2\Omega}j^a[\mathring{\delta}\boldsymbol{g}]$							

[a] This result only holds if the null fluid of gravitons that arises in geometric optics is either completely ingoing or outgoing at null infinity; see the discussion near the end of section 3.4.3 for more details.

In this chapter, following [83], we consider four conserved currents for linearized gravity, denoted ${}_{2\mathring{\mathcal{C}}}j^a[\mathring{\delta}\boldsymbol{g}]$, ${}_{2\mathring{\mathcal{D}}}j^a[\mathring{\delta}\boldsymbol{g}]$, and ${}_{\pm 2\Omega}j^a[\mathring{\delta}\boldsymbol{g}]$. These conserved currents generalize the Carter constant in Kerr, in the sense that each of their charges reduce to a sum of some positive power of the Carter constants of gravitons in the geometric optics limit. Moreover, we show that these currents have the further property that their fluxes at null infinity and the horizon are finite for well-behaved solutions that describe radiation. While these currents themselves are new to [83], their construction involves symmetry operators which have been studied extensively in the literature (see, for example, [177, 54, 10]). A brief summary of their properties, along with those of a few additional currents which do *not* share the property of possessing finite fluxes, is given in table 3.1.

The organization of this chapter is as follows. Section 3.1 is a review of the theory of linearized gravity in Kerr, using both the spinor and Newman-Penrose formalisms, and fixes conventions which we use throughout. It also reviews the Teukolsky formalism and separation of variables in the Kerr spacetime. Section 3.2 defines symmetry operators, which are the maps from the space of solutions into itself, such as the operator ${}_0\mathcal{D}$ in equation (3.2) above. We give particular examples of symmetry operators for linearized gravity in Kerr, and show how they act on expansions that arise in the Teukolsky formalism. In section 3.3, we first define the symplectic product, a generalization of the Klein-Gordon inner product used in the scalar case, which we then use to generate the conserved currents in table 3.1. In section 3.4, we review the geometric optics limit of solutions in linearized gravity on a curved background and use it to deduce the limits of currents defined in section 3.3. In section 3.5, we compute fluxes of these currents through the horizon and null infinity.

We use the following conventions in this chapter: we follow most texts on spinors by using the $(+,-,-,-)$ sign convention for the metric and bars to denote complex conjugation. Moreover, we also use the convention for the curvature tensor of Penrose and Rindler [129, 130], which is notably *opposite* that of Wald [176]. For any linear operator $T_{a_1\cdots a_p}{}^{b_1\cdots b_q}$ which maps tensors of rank q to those of rank p, we write $T_{a_1\cdots a_p}{}^{b_1\cdots b_q}S_{b_1\cdots b_q}$ as $\boldsymbol{T}\cdot\boldsymbol{S}$ when indices have been removed. Finally, we will leave explicit the soldering forms $\sigma_a{}^{AA'}$ which form the isomorphism between the tangent vector space and the space of Hermitian spinors [129].

Finally, note that many results in this chapter have analogues in electromagnetism. For brevity, we do not discuss the electromagnetic case, which is discussed in depth in [82], which is *not* contained in this thesis.

3.1 Kerr Perturbation Theory

3.1.1 Spinors

In this chapter, we will be using a combination of the spinor and Newman-Penrose formalisms in order to describe linearized gravity about some arbitrary vacuum solution of the Einstein equations. In general, we follow the notation of Penrose and Rindler [129, 130].[1] The spinor formalism is particularly convenient in Kerr, since not only is there a rank two Killing tensor K_{ab}, but also a rank two symmetric spinor ζ_{AB} which satisfies the *Killing spinor equation* [130]:

$$\nabla^{A'}{}_{(A}\zeta_{BC)} = 0. \tag{3.6}$$

[1] For a less opaque introduction to spinors, I recommend chapter 2 of Stewart [152].

This Killing spinor is related to the Killing-Yano tensor f_{ab} by

$$f_{ab} = \sigma_a{}^{AA'}\sigma_b{}^{BB'}\left(i\epsilon_{A'B'}\zeta_{AB} - i\epsilon_{AB}\bar{\zeta}_{A'B'}\right). \tag{3.7}$$

This Killing spinor generates the related *conformal Killing tensor* Σ_{ab} given by

$$\Sigma_{ab} \equiv \sigma_a{}^{AA'}\sigma_b{}^{BB'}\zeta_{AB}\bar{\zeta}_{A'B'} \equiv \frac{1}{2}K_{ab} - \frac{1}{4}\operatorname{Re}\left[\zeta_{CD}\zeta^{CD}\right]g_{ab}, \tag{3.8}$$

which we use to define our Killing tensor K_{ab} [179]; note that this is equivalent to $K_{ab} = f_{ac}f^c{}_b$. Note that, given a Killing spinor ζ_{AB}, equation (3.8) fixes the ambiguity in K_{ab}, which is otherwise only defined only up to terms of the form λg_{ab}, for constant λ, or up to terms that are products of Killing vectors.

Vacuum Petrov type D spacetimes (such as Kerr) possess a Killing spinor intimately connected with the Weyl spinor Ψ_{ABCD} [179], the symmetric spinor constructed from the Weyl tensor:

$$C_{abcd} \equiv \sigma_a{}^{AA'}\sigma_b{}^{BB'}\sigma_c{}^{CC'}\sigma_d{}^{DD'}\left(\epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{A'B'C'D'} + \epsilon_{A'B'}\epsilon_{C'D'}\Psi_{ABCD}\right). \tag{3.9}$$

Since Ψ_{ABCD} is symmetric, it can be written as a symmetric product of four spinors

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}. \tag{3.10}$$

For spacetimes of Petrov type D, there is a choice of these spinors such that $\alpha_A = \beta_A$ and $\gamma_A = \delta_A$ (this is one of many equivalent definitions of a type D spacetime). Normalizing α_A and γ_A to be a spin basis (o, ι) (that is, setting $o_A\iota^A = 1$), one finds

$$\Psi_{ABCD} = 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)}. \tag{3.11}$$

We are using the following notation for contractions of spinors with a given spin basis [129]: given a symmetric spinor field $S_{B_1\cdots B_n}$ and a spin basis (o, ι), we define (for any integer i with $0 \leq i \leq n$)

$$S_i = S_{B_1\cdots B_n}\iota^{B_1}\cdots\iota^{B_i}o^{B_{i+1}}o^{B_n}. \tag{3.12}$$

Thus, in equation (3.11), Ψ_2 means the Weyl scalar $\Psi_{ABCD}\iota^A\iota^B o^C o^D$. The spin basis (o, ι) is called a *principal spin basis* for the Weyl spinor if it satisfies equation (3.11). On a principal spin basis, one can show that there exists a Killing spinor ζ_{AB} defined by

$$\zeta_{AB} \equiv \zeta o_{(A}\iota_{B)}, \tag{3.13}$$

where $\zeta\sqrt[3]{\Psi_2}$ is constant [179]. For the remainder of the chapter, we will restrict ourselves (generally) to a principal spin basis of the background Weyl spinor.

With these definitions in hand, we turn to the construction of linearized gravity in Kerr. We fix the background Kerr metric g_{ab}, and consider a one-parameter family of metrics $g_{ab}(\lambda)$, with $g_{ab}(0) = g_{ab}$. In general, we will use a notational convention where, for any quantity Q, $Q(\lambda)$ will denote the quantity at an arbitrary value of λ, and Q without an argument will denote $Q(0)$, the background value. The *linearization* δQ of $Q(\lambda)$ is defined by[2]

$$\delta Q = \left.\frac{\mathrm{d}Q}{\mathrm{d}\lambda}\right|_{\lambda=0}. \tag{3.14}$$

The linearized Einstein equations take the form

$${}_2\mathcal{E}^{abcd}\delta g_{cd} = 8\pi\delta T^{ab}, \tag{3.15}$$

where

$${}_2\mathcal{E}^{abcd} \equiv -\nabla^{(c}g^{d)(a}\nabla^{b)} + \frac{1}{2}(g^{cd}\nabla^{(a}\nabla^{b)} + g^{ac}g^{bd}\Box) - \frac{1}{2}g^{ab}(g^{cd}\Box - \nabla^{(c}\nabla^{d)}). \tag{3.16}$$

is the linearized Einstein operator and δT^{ab} is the linearized stress-energy tensor. Here the covariant derivative ∇_a is that associated with g_{ab}; the covariant derivative associated with $g_{ab}(\lambda)$ is denoted $\nabla_a(\lambda)$. The prefixed subscript 2 in ${}_2\mathcal{E}^{abcd}$ refers to the fact that linearized gravity is a spin-2 field.

To describe linearized perturbations using spinors, we consider the following quantity:

$$(\delta g)_{AA'BB'} \equiv \sigma^a{}_{AA'}\sigma^b{}_{BB'}\delta g_{ab}. \tag{3.17}$$

Note that this is *not* the variation of a spinor; we are performing the variation *first*, and then computing a spinor field using the soldering forms $\sigma^a{}_{AA'}$ that are associated with the background spacetime[3]. In general, the placement of parentheses around a quantity that we are varying implies that we take the variation first, and then perform the operation, such as raising or lowering indices: for example, $(\delta g)^{ab} = g^{ac}g^{bd}\delta g_{cd}$, whereas δg^{ab} would be the variation of the raised metric, and in fact $\delta g^{ab} = -(\delta g)^{ab}$.

[2] We are using δ, instead of the more conventional δ, in order to avoid confusion with the Newman-Penrose operator δ. Note that we used this symbol for a variation throughout this thesis, for consistency.

[3] We note that there have been recent developments on a variational formalism for spinors [25] which we will not be using. We instead follow the traditional approach of [129].

In a similar manner, one can define a spinor $(\delta\Psi)_{ABCD}$ that is frequently called the perturbed Weyl spinor [129] (although it is also not the variation of a spinor), again using the background soldering forms:

$$(\delta\Psi)_{ABCD} \equiv \frac{1}{4}\sigma^a{}_{AE'}\sigma^b{}_B{}^{E'}\sigma^c{}_{CF'}\sigma^d{}_D{}^{F'}\delta C_{abcd}. \tag{3.18}$$

Using the form of the perturbed Riemann tensor, one finds that [129]

$$(\delta\Psi)_{ABCD} = \frac{1}{2}\nabla^{A'}{}_{(C}\nabla^{B'}{}_D(\delta g)_{AB)A'B'} + \frac{1}{4}(\delta g)_e{}^e\Psi_{ABCD}. \tag{3.19}$$

The equations of motion for the perturbed Weyl spinor are derived from the Bianchi identity, and are [129]

$$\begin{aligned}\nabla^{AA'}(\delta\Psi)_{ABCD} = \frac{1}{2}(\delta g)^{EFA'B'}\nabla_{BB'}\Psi_{EFCD} - \Psi_{EF(BC}\nabla_{D)}{}^{B'}(\delta g)^{EFA'}{}_{B'} \\ - \frac{1}{2}\Psi_{EF(BC}\nabla^{EB'}(\delta g)_{D)}{}^{FA'}{}_{B'}.\end{aligned} \tag{3.20}$$

Thus, the equations of motion depend explicitly on the metric perturbation as well as the perturbed Weyl spinor. Note further that equation (3.20) reduces to the spin-2 massless spinor field equation $\nabla^{AA'}(\delta\Psi)_{ABCD} = 0$ only when the manifold is conformally flat ($\Psi_{ABCD} = 0$).

The perturbed Weyl spinor, moreover, is not gauge invariant: under a gauge transformation $\delta g_{ab} \to \delta g_{ab} + 2\nabla_{(a}\xi_{b)}$ [129],

$$(\delta\Psi)_{ABCD} \to (\delta\Psi)_{ABCD} + \xi^{EE'}\nabla_{E'(A}\Psi_{BCD)E} + 2\Psi_{E(ABC}\nabla_{D)E'}\xi^{EE'}. \tag{3.21}$$

For type D spacetimes, however, $(\delta\Psi)_0$ and $(\delta\Psi)_4$ *are* gauge invariant, and they are the pieces that correspond to gravitational radiation [153]. Moreover, as is well known, the equations of motion for $(\delta\Psi)_0$ and $(\delta\Psi)_4$ can be "decoupled" from those for $(\delta\Psi)_1$, $(\delta\Psi)_2$, and $(\delta\Psi)_3$, and each other [162], as we will discuss in section 3.1.3. It suffices to use either $(\delta\Psi)_0$ or $(\delta\Psi)_4$ to describe a generic, well-behaved perturbation, up to $l = 0, 1$ modes [175], and therefore we can describe such perturbations in terms of gauge invariant variables.

3.1.2 Spin coefficients

We will also be using the Newman-Penrose notation: given a spin basis (o, ι), the null basis $\{l^a, n^a, m^a, \bar{m}^a\}$ is defined by

$$l^a = \sigma^a{}_{AA'}o^A\bar{o}^{A'}, \quad n^a = \sigma^a{}_{AA'}\iota^A\bar{\iota}^{A'}, \quad m^a = \sigma^a{}_{AA'}o^A\bar{\iota}^{A'}, \tag{3.22}$$

such that

$$g_{ab} = 2(l_{(a}n_{b)} - m_{(a}\bar{m}_{b)}). \tag{3.23}$$

Using these four vectors, one can define the Newman-Penrose operators by $D = l^a\nabla_a$, $\mathbb{\Delta} = n^a\nabla_a$, and $\delta = m^a\nabla_a$, as well as the twelve spin coefficients via the following eight equations:

$$\begin{aligned}
Do_A &= \epsilon o_A - \kappa\iota_A, & D\iota_A &= \pi o_A - \epsilon\iota_A,\\
\mathbb{\Delta} o_A &= \gamma o_A - \tau\iota_A, & \mathbb{\Delta}\iota_A &= \nu o_A - \gamma\iota_A,\\
\delta o_A &= \beta o_A - \sigma\iota_A, & \delta\iota_A &= \mu o_A - \beta\iota_A,\\
\bar{\delta} o_A &= \alpha o_A - \rho\iota_A, & \bar{\delta}\iota_A &= \lambda o_A - \alpha\iota_A.
\end{aligned} \tag{3.24}$$

The five Weyl scalars Ψ_0, Ψ_1, Ψ_2, Ψ_3, and Ψ_4, in Newman-Penrose notation, take the form [121]

$$\Psi_i = -C_{abcd}\begin{cases} l^a m^b l^c m^d & i=0\\ l^a n^b l^c m^d & i=1\\ \frac{1}{2} l^a n^b (l^c n^d - m^c \bar{m}^d) & i=2\\ l^a n^b \bar{m}^c n^d & i=3\\ n^a \bar{m}^b n^c \bar{m}^d & i=4 \end{cases}. \tag{3.25}$$

A null tetrad such that $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and $\Psi_2 \neq 0$, for a Petrov type D spacetime, is called a *principal tetrad* (as it is a tetrad associated with a principal spin basis).

Furthermore, at certain points throughout this chapter, we will be using the notion of $'$ and $*$ transformations to simplify the presentation. These are defined by replacing, in some expression, the members of the spin basis via the following rules:[4]

$$\begin{aligned}
{}' : o_A \mapsto i\iota_A, \quad \iota_A \mapsto io_A, \quad \bar{o}_{A'} \mapsto -i\bar{\iota}_{A'}, \quad \bar{\iota}_{A'} \mapsto -i\bar{o}_{A'},\\
* : o_A \mapsto o_A, \quad \iota_A \mapsto \iota_A, \quad \bar{o}_{A'} \mapsto -\bar{\iota}_{A'}, \quad \bar{\iota}_{A'} \mapsto -\bar{o}_{A'}.
\end{aligned} \tag{3.26}$$

The $'$ and $*$ transformations elucidate certain symmetries that appear in Newman-Penrose notation. The $'$ transformation, which merely switches $l^a \leftrightarrow n^a$ and $m^a \leftrightarrow \bar{m}^a$, is particularly important in Kerr, since it preserves (o, ι) as a principal spin basis. As an example, applying the transformations

[4] Note that this definition of $'$ and $*$ is that of Stewart [152], and disagrees with that of Penrose and Rindler [129] by a sign in the $*$ operation. This does not qualitatively change any of the results of this discussion.

to equation (3.24) yields

$$\begin{aligned} \epsilon' &= -\gamma, \quad \kappa' = -\nu, \quad \pi' = -\tau, \\ \beta' &= -\alpha, \quad \sigma' = -\lambda, \quad \mu' = -\rho, \\ \epsilon^* &= -\beta, \quad \kappa^* = -\sigma, \quad \pi^* = -\mu, \\ \gamma^* &= -\alpha, \quad \tau^* = -\rho, \quad \nu^* = -\lambda. \end{aligned} \tag{3.27}$$

As another example, consider the following equations, in Newman-Penrose notation, that the scalar ζ obeys in Kerr:

$$D\zeta = -\zeta\rho, \qquad \Delta\zeta = \zeta\mu, \qquad \delta\zeta = -\zeta\tau, \qquad \bar{\delta}\zeta = \zeta\pi. \tag{3.28}$$

The second equation can be derived from the first via a $'$ transformation, and likewise the fourth from the third, while the third follows from the first via a $*$ transformation. In the future, we will only list one of the equations, and specify that the others can be obtained by the appropriate transformations.

3.1.3 The Teukolsky equation

The Teukolsky formalism is a choice of variables for test fields in Kerr such that the equations of motion decouple, yielding equations that describe radiation, and furthermore, as we will discuss later in this section, separate in Boyer-Lindquist coordinates. It builds off of the Newman-Penrose formalism: in the case of linearized gravity, the variables involve variations of the Weyl scalars. Note that, taking variations of the Weyl scalars, we find that

$$\eth\Psi_0 = (\eth\Psi)_0, \qquad \eth\Psi_4 = (\eth\Psi)_4. \tag{3.29}$$

On the left-hand sides of these equations, there is a variation of the null tetrad as well as the Weyl tensor; on the right, only the Weyl tensor is varied, according to equation (3.18). Note that equation (3.29) only holds for $\eth\Psi_0$ and $\eth\Psi_4$, and only because the background is type D, as the tetrad is varied when varying equation (3.25). This result is rather convenient, since we will have reason to use $\eth\Psi_0$ and $(\eth\Psi)_0$, for example, interchangeably.

The choice of variables that are employed here are the so-called "master variables" ${}_s\Omega$, defined by [162]

$$
{}_s\Omega \equiv \begin{cases} \zeta^4 \delta\Psi_4 & s=-2 \\ \Phi & s=0 \\ \delta\Psi_0 & s=2 \end{cases} . \tag{3.30}
$$

The value of s is known as the *spin-weight* of the particular variable. Moreover, for $s>0$, one can write these variables in terms of an operator ${}_s\boldsymbol{M}$, which maps the original field (such as the metric perturbation δg_{ab}) to the corresponding master variable ${}_s\Omega$. For example, for $|s|=2$,

$$
{}_s\Omega = {}_sM^{ab}\delta g_{ab}. \tag{3.31}
$$

From equations (3.18), (3.30), and (3.31) (see, for example, [54]),

$$
\begin{aligned}
{}_2M^{ab} = \frac{1}{2}\Big\{ &\big[(D-\bar\rho-3\epsilon+\bar\epsilon)(\delta+2\bar\pi-2\beta)+(\delta+\bar\pi-3\beta-\bar\alpha)(D-2\bar\rho-2\epsilon)\big]l^a m^b \\
&-(\delta+\bar\pi-3\beta-\bar\alpha)(\delta+\bar\pi-2\beta-2\bar\alpha)l^a l^b \\
&-(D-\bar\rho-3\epsilon+\bar\epsilon)(D-\bar\rho-2\epsilon+2\bar\epsilon)m^a m^b\Big\},
\end{aligned} \tag{3.32a}
$$

$$
\begin{aligned}
{}_{-2}M^{ab} = \frac{\zeta^4}{2}\Big\{ &\big[(\Delta+\bar\mu+3\gamma-\bar\gamma)(\bar\delta-2\bar\tau+2\alpha)+(\bar\delta-\bar\tau+3\alpha+\bar\beta)(\Delta+2\bar\mu+2\gamma)\big]n^a \bar m^b \\
&-(\bar\delta-\bar\tau+3\alpha+\bar\beta)(\bar\delta-\bar\tau+2\alpha+2\bar\beta)n^a n^b \\
&-(\Delta+\bar\mu+3\gamma-\bar\gamma)(\Delta+\bar\mu+2\gamma-2\bar\gamma)\bar m^a \bar m^b\Big\}.
\end{aligned} \tag{3.32b}
$$

In terms of these variables, and in a type D spacetime, the equations of motion for the scalar field Φ ($s=0$) and linearized gravity ($s=\pm2$) may be written in the form [162]

$$
{}_s\Box\, {}_s\Omega = 8\pi\, {}_s\boldsymbol{\tau}\cdot {}_{|s|}\boldsymbol{T}, \tag{3.33}
$$

which is known as the *Teukolsky equation*. Here, ${}_s\Box$ is a second-order differential operator (the *Teukolsky operator*) that equals, for $s\ge 0$,

$$
\begin{aligned}
{}_s\Box = 2\{&[D-(2s-1)\epsilon+\bar\epsilon-2s\rho-\bar\rho](\Delta-2s\gamma+\mu) \\
&-[\delta-\bar\alpha-(2s-1)\beta-2s\tau+\bar\pi](\bar\delta-2s\alpha+\pi)-(2s-1)(s-1)\Psi_2\},
\end{aligned} \tag{3.34a}
$$

$$
\begin{aligned}
{}_{-s}\Box = 2\{&[\Delta+(2s-1)\gamma-\bar\gamma+\bar\mu][D+2s\epsilon+(2s-1)\rho] \\
&-[\bar\delta+(2s-1)\alpha+\bar\beta-\bar\tau][\delta+2s\beta+(2s-1)\tau]-(2s-1)(s-1)\Psi_2\}.
\end{aligned} \tag{3.34b}
$$

On the right-hand side of equation (3.33), ${}_s\boldsymbol{\tau}$ is an operator which converts ${}_s\boldsymbol{T}$, the source term for the equations of motion (for example, ${}_2T^{ab}$ is the stress-energy tensor δT^{ab}), into the source term for the Teukolsky equation (3.33). For example, one choice of ${}_{\pm 2}\tau_{ab}$ is given by inspection of equations (2.13) and (2.15) of [162]:

$$\begin{aligned} {}_2\tau_{ab} = {} & \left[(\delta + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)l_{(a|} - (D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})m_{(a|}\right] \\ & \times \left[(D - \epsilon + \bar{\epsilon} - \bar{\rho})m_{|b)} - (\delta + \bar{\pi} - \bar{\alpha} - \beta)l_{|b)}\right], \end{aligned} \tag{3.35a}$$

$$\begin{aligned} {}_{-2}\tau_{ab} = {} & \zeta^4 \left[(\mathbb{A} + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})\bar{m}_{(a|} - (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)n_{(a|}\right] \\ & \times \left[(\bar{\delta} - \bar{\tau} + \bar{\beta} + \alpha)n_{|b)} - (\mathbb{A} + \gamma - \bar{\gamma} + \bar{\mu})\bar{m}_{|b)}\right]. \end{aligned} \tag{3.35b}$$

A freedom in ${}_{\pm 2}\tau_{ab}$ is discussed in section 3.2.3 below. One can also rewrite Teukolsky's original result as an operator equation [177], as we will find useful in section 3.2.2. In terms of ${}_s\boldsymbol{M}$,

$${}_s\boldsymbol{\tau} \cdot {}_{|s|}\boldsymbol{\mathcal{E}} = {}_s\Box\, {}_s\boldsymbol{M}, \tag{3.36}$$

where, for $|s| = 2$, ${}_{|s|}\boldsymbol{\mathcal{E}}$ is the linearized Einstein operator (3.16). Applying equation (3.36) to a metric perturbation and using equation (3.31) and the linearized Einstein equation (3.15) yields the Teukolsky equation (3.33) for $|s| = 2$. Since all of the operations just described are $\mathbb{C}$-linear, equation (3.36) holds for complexified metric perturbations as well.

So far, we have not tied our discussion to a particular coordinate system, nor a particular tetrad (other than enforcing that we use a principal null tetrad), since we have only required the background metric to be Petrov type D. We now work in Kerr, and in Boyer-Lindquist coordinates (t, r, θ, ϕ), where the metric takes the form [note the sign change relative to equation (2.1)]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) - (r^2 + a^2)\sin^2\theta \mathrm{d}\phi^2 - \frac{2Mr}{\Sigma}\left(a\sin^2\theta\mathrm{d}\phi - \mathrm{d}t\right)^2, \tag{3.37}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta = |\zeta|^2$, and where we have chosen

$$\zeta = r - ia\cos\theta. \tag{3.38}$$

This choice of ζ has the property that $t^a \equiv (\partial_t)^a$ can be defined in terms of ζ_{AB} [130]:

$$t^{AA'} = -\frac{2}{3}\nabla_B{}^{A'}\zeta^{AB}. \tag{3.39}$$

Using the Kinnersley tetrad (a principal tetrad of the background Weyl tensor), which is given by

$$\boldsymbol{l} = \frac{(r^2+a^2)\partial_t + a\partial_\phi}{\Delta} + \partial_r, \quad \boldsymbol{n} = \frac{(r^2+a^2)\partial_t + a\partial_\phi}{2\Sigma} - \frac{\Delta}{2\Sigma}\partial_r, \\ \boldsymbol{m} = \frac{1}{\sqrt{2}\bar{\zeta}}\left(ia\sin\theta\partial_t + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right), \tag{3.40}$$

we find that $\Psi_2 = -M/\zeta^3$. Furthermore, the non-zero spin coefficients are given by

$$\rho = -\frac{1}{\zeta}, \quad \mu = -\frac{\Delta}{2\Sigma\zeta}, \quad \gamma = \mu + \frac{r-M}{2\Sigma}, \\ \beta = \frac{\cot\theta}{2\sqrt{2}\bar{\zeta}}, \quad \pi = \alpha + \bar{\beta} = \frac{ia}{\sqrt{2}\zeta^2}\sin\theta, \quad \tau = -\frac{ia}{\sqrt{2}\Sigma}\sin\theta. \tag{3.41}$$

We now review how the source-free version of the Teukolsky equation (3.33) separates in these coordinates. Consider, for integers s and n, the operators [163, 52]

$$\mathscr{D}_n = \partial_r + \frac{r^2+a^2}{\Delta}\partial_t + \frac{a}{\Delta}\partial_\phi + 2n\frac{r-M}{\Delta}, \quad \mathscr{L}_s = \partial_\theta - i\left(a\sin\theta\partial_t + \frac{1}{\sin\theta}\partial_\phi\right) + s\cot\theta. \tag{3.42}$$

Note that these operators satisfy

$$\Delta^{-m}\mathscr{D}_n\Delta^m = \mathscr{D}_{n+m}, \qquad \sin^{-r}\theta\mathscr{L}_s\sin^r\theta = \mathscr{L}_{r+s}. \tag{3.43}$$

We also define the operators $\mathscr{D}_n^+$ and $\mathscr{L}_s^+$ by taking $\mathscr{D}_n$ and $\mathscr{L}_s$ and setting $\partial_t \to -\partial_t$ and $\partial_\phi \to -\partial_\phi$; note that $\mathscr{L}_s^+ = \overline{\mathscr{L}_s}$ [5]. Equations analogous to equations (3.43) hold for $\mathscr{D}_n^+$ and $\mathscr{L}_s^+$. We will also need a way to express these operators in terms of Newman-Penrose operators; using equations (3.40) and (3.41), we find

$$\mathscr{L}_s = \sqrt{2}\zeta\left(\bar{\delta} + 2s\bar{\beta}\right), \qquad \mathscr{D}_n = D + 2n\rho\mu^{-1}(\gamma-\mu), \qquad \mathscr{D}_n^+ = -\rho\mu^{-1}[\mathbb{\Delta} - 2n(\gamma-\mu)]. \tag{3.44}$$

Note that these formulae are only valid for the Kinnersley tetrad. For real frequencies ω and integers m, we further define operators $\mathscr{D}_{nm\omega}$ and $\mathscr{L}_{sm\omega}$ by the requirement that, for any function $f(r,\theta)$,

$$\mathscr{D}_n\left[e^{i(m\phi-\omega t)}f(r,\theta)\right] \equiv e^{i(m\phi-\omega t)}\mathscr{D}_{nm\omega}f(r,\theta), \quad \mathscr{L}_s\left[e^{i(m\phi-\omega t)}f(r,\theta)\right] \equiv e^{i(m\phi-\omega t)}\mathscr{L}_{sm\omega}f(r,\theta). \tag{3.45}$$

These equations yields the formulae

$$\mathscr{D}_{nm\omega} \equiv \partial_r + \frac{iK_{m\omega}}{\Delta} + 2n\frac{r-M}{\Delta}, \quad \mathscr{L}_{sm\omega} \equiv \partial_\theta + Q_{m\omega} + s\cot\theta, \tag{3.46}$$

[5] Note that here, and below, our definition of the complex conjugate $\overline{\mathcal{O}}$ of an operator $\mathcal{O}$ is $\overline{\mathcal{O}}(f) = \overline{\mathcal{O}(\bar{f})}$, where f is the argument of this operator. This is consistent with the standard notation for the Newman-Penrose operator $\bar{\delta}$.

where

$$K_{m\omega} \equiv am - \omega(r^2 + a^2), \quad Q_{m\omega} \equiv m \csc\theta - a\omega \sin\theta \tag{3.47}$$

(note that the conventions for $K_{m\omega}$ in [52] and [162] differ by a sign; here, we use the convention of [52]).

The operator on the left-hand side of the Teukolsky equation (3.33) takes the following simple form:

$${}_s\Box = {}_s\mathcal{R} + {}_s\mathcal{S}, \tag{3.48}$$

where

$${}_s\mathcal{R} \equiv \begin{cases} \Delta\mathscr{D}_1\mathscr{D}_s^+ - 2(2s-1)r\partial_t & s \geq 0 \\ \Delta\mathscr{D}_{1+s}^+\mathscr{D}_0 - 2(2s+1)r\partial_t & s \leq 0 \end{cases}, \tag{3.49a}$$

$${}_s\mathcal{S} \equiv \begin{cases} \mathscr{L}_{1-s}^+\mathscr{L}_s + 2i(2s-1)a\cos\theta\partial_t & s \geq 0 \\ \mathscr{L}_{1+s}\mathscr{L}_{-s}^+ + 2i(2s+1)a\cos\theta\partial_t & s \leq 0 \end{cases}, \tag{3.49b}$$

where it can be readily shown that either the top or bottom lines of equations (3.49a) and (3.49b) yield equal results for $s = 0$; that is, ${}_{+0}\mathcal{R} = {}_{-0}\mathcal{R}$ and ${}_{+0}\mathcal{S} = {}_{-0}\mathcal{S}$. Note that ${}_s\mathcal{R}$ is a differential operator that only depends on r, t, and ϕ, while ${}_s\mathcal{S}$ only depends on θ, t, and ϕ. As such, it is clear that the *sourceless* Teukolsky equation (3.33) separates in r and θ, and so one can write [162]

$${}_s\Omega(t, r, \theta, \phi) = \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=|s|}^{\infty} \sum_{|m|\leq l} {}_s\widehat{\Omega}_{lm\omega}(r)\, {}_s\Theta_{lm\omega}(\theta) e^{i(m\phi - \omega t)}. \tag{3.50}$$

Inserting this expansion into the sourceless Teukolsky equation (3.33), followed by using equations (3.48), (3.49), (3.43), and (3.45), one finds that (for $s \geq 0$), the functions ${}_{\pm s}\widehat{\Omega}_{lm\omega}$ and ${}_{\pm s}\Theta_{lm\omega}$ satisfy [52]

$$\left[\mathscr{L}_{(1-s)(\mp m)(\mp\omega)}\mathscr{L}_{s(\pm m)(\pm\omega)} \pm 2(2s-1)\omega a\cos\theta\right] {}_{\pm s}\Theta_{lm\omega} = -\, {}_{\pm s}\lambda_{lm\omega}\, {}_{\pm s}\Theta_{lm\omega}, \tag{3.51a}$$

$$\left[\Delta\mathscr{D}_{(1-s)(\pm m)(\pm\omega)}\mathscr{D}_{0(\mp m)(\mp\omega)} \pm 2i(2s-1)\omega r\right] \Delta^{(s\pm s)/2}\, {}_{\pm s}\widehat{\Omega}_{lm\omega} = \Delta^{(s\pm s)/2}\, {}_{\pm s}\lambda_{lm\omega}\, {}_{\pm s}\widehat{\Omega}_{lm\omega}, \tag{3.51b}$$

where ${}_{\pm s}\lambda_{lm\omega}$ is a separation constant. This constant reduces to $(l+s)(l-s+1) = l(l+1) - s(s-1)$ in the Schwarzschild limit [136, 52].

The functions ${}_s\Theta_{lm\omega}$ are regular solutions to a Sturm-Liouville problem on $[0, \pi]$ with eigenvalues ${}_s\lambda_{lm\omega}$. Thus, there is only one solution for each value of l, m, and ω, up to scaling. Note, moreover, that the differential operator on the left-hand side of equation (3.51a) commutes with the following three operations: complex conjugation, $(s, m, \omega) \to (-s, -m, -\omega)$, and $(s, \theta) \to (-s, \pi - \theta)$. As such, we can simultaneously diagonalize this operator with each of these operations, choosing ${}_s\lambda_{lm\omega}$ and ${}_s\Theta_{lm\omega}$ to be real, as well as choosing

$$ {}_s\Theta_{lm\omega}(\theta) = (-1)^{m+s} {}_{-s}\Theta_{l(-m)(-\omega)}(\theta), \qquad {}_s\Theta_{lm\omega}(\pi - \theta) = (-1)^{l+m} {}_{-s}\Theta_{lm\omega}(\theta) \tag{3.52} $$

(a convention which is used by [76]), as well as

$$ {}_s\lambda_{lm\omega} = {}_{-s}\lambda_{lm\omega} = {}_s\lambda_{l(-m)(-\omega)}. \tag{3.53} $$

Finally, the scaling freedom in ${}_s\Theta_{lm\omega}$ is fixed by imposing the following normalization condition [162]

$$ \int_0^\pi {}_s\Theta_{lm\omega}(\theta)\, {}_s\Theta_{l'm\omega}(\theta) \sin\theta \mathrm{d}\theta = \delta_{ll'}. \tag{3.54} $$

The functions

$$ {}_sY_{lm\omega}(\theta, t, \phi) \equiv e^{i(m\phi - \omega t)} {}_s\Theta_{lm\omega}(\theta) \tag{3.55} $$

are the so-called *spin-weighted spheroidal harmonics*, and (for a given s) are orthogonal for different l, m, and ω. The coefficients ${}_s\widehat{\Omega}_{lm\omega}$ can therefore be thought of as a coefficients in an expansion in spin-weighted spheroidal harmonics.

We now define another expansion for ${}_s\Omega$, subtly different from that in equation (3.50), which results in a convenient way of expanding $\overline{{}_s\Omega}$ as well. To do so, note that the differential operator on the right-hand side of equation (3.51b) commutes with taking $(m, \omega) \to (-m, -\omega)$ followed by complex conjugation. As such, we can construct two linearly independent solutions labeled by $p = \pm 1$ [their eigenvalue under this operation, multiplied by a conventional factor of $(-1)^{m+s}$]:

$$ {}_s\widehat{\Omega}_{lm\omega p}(r) \equiv \frac{1}{2}\left[{}_s\widehat{\Omega}_{lm\omega}(r) + p(-1)^{m+s}\overline{{}_s\widehat{\Omega}_{l(-m)(-\omega)}(r)} \right], \tag{3.56} $$

and so

$$ {}_s\widehat{\Omega}_{lm\omega}(r) = \sum_{p=\pm 1} {}_s\widehat{\Omega}_{lm\omega p}(r). \tag{3.57} $$

It is occasionally more convenient to re-express the expansion (3.50) in terms of ${}_s\widehat{\Omega}_{lm\omega p}(r)$, instead of ${}_s\widehat{\Omega}_{lm\omega}(r)$:

$$ {}_s\Omega(t, r, \theta, \phi) = \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=|s|}^{\infty} \sum_{|m|\le l} \sum_{p=\pm 1} e^{i(m\phi - \omega t)} {}_s\Theta_{lm\omega}(\theta)\, {}_s\widehat{\Omega}_{lm\omega p}(r). \tag{3.58} $$

A simple consequence of equations (3.52) and (3.56) is that

$$\overline{{}_s\Omega(t,r,\theta,\phi)} = \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=|s|}^{\infty} \sum_{|m|\leq l} \sum_{p=\pm 1} p e^{i(m\phi-\omega t)} \, {}_{-s}\Theta_{lm\omega}(\theta) \, {}_s\widehat{\Omega}_{lm\omega p}(r), \tag{3.59}$$

and so this is a convenient expansion of the complex conjugate of the master variables. Note, however, that these expansions are different in status from the expansion (3.50), as the coefficients in this expansion must satisfy

$$\overline{{}_s\widehat{\Omega}_{l(-m)(-\omega)p}(r)} = p(-1)^{m+s} \, {}_s\widehat{\Omega}_{lm\omega p}(r). \tag{3.60}$$

3.2 Symmetry Operators in the Kerr Spacetime

As defined by Kalnins, McLenaghan, and Williams [106], a *symmetry operator* is an $\mathbb{R}$-linear operator that maps the space of solutions to the equations of motion, which must be linear, into itself. For the space of complexified solutions to real equations of motion, there exists a trivial symmetry operator mapping solutions to their complex conjugates. In his original paper, Carter constructed the symmetry operator for scalar fields in equation (3.2), which commutes with the d'Alembertian [49]. If an operator commutes with the operators in the sourceless equations of motion, then it must be a symmetry operator: if a field ϕ satisfies $\mathcal{L}\phi = 0$, and $[\mathcal{D}, \mathcal{L}] = 0$, then

$$\mathcal{L}\mathcal{D}\phi = \mathcal{D}\mathcal{L}\phi = 0, \tag{3.61}$$

and so $\mathcal{D}\phi$ is a solution. Lie derivatives with respect to Killing vectors are examples of symmetry operators which commute with the equations of motion. Further examples of symmetry operators can be created by composing symmetry operators associated with Killing vectors, but these are, in a sense, "reducible".

In this section we review two classes of *irreducible* symmetry operators that appear in the Kerr spacetime: those that derive from separation of variables, and those that arise from taking the adjoint of the Teukolsky equation. Note that, recently, additional symmetry operators have been discussed in the Kerr spacetime [10], which we do not discuss in this chapter.

3.2.1 Separation of variables

The first class of symmetry operators we consider is associated with the separability of the underlying equations of motion. To see that there is always a symmetry operator associated with separability, consider as an example the following partial differential equation (in two variables x, y):

$$\mathcal{L}\phi \equiv \left[\mathcal{X}(x, \partial_x, \ldots) + \mathcal{Y}(y, \partial_y, \partial_y^2, \ldots)\right]\phi = 0, \tag{3.62}$$

for some differential operators $\mathcal{X}$ and $\mathcal{Y}$. Since $\mathcal{X}$ only depends upon x and $\mathcal{Y}$ only depends upon y, $\mathcal{X}$ and $\mathcal{Y}$ must commute. Moreover, $\mathcal{L} = \mathcal{X} + \mathcal{Y}$, and so $\mathcal{X}$ and $\mathcal{Y}$ must both commute with $\mathcal{L}$, and so $\mathcal{X}$ and $\mathcal{Y}$ are symmetry operators. In addition, if there are additional variables $z_1, \ldots, z_n$, and $\mathcal{X}$ and $\mathcal{Y}$ only depend on derivatives with respect to these variables, then this argument still holds.

Irreducible symmetry operators arise in Kerr, similarly, via a separation of variables argument. As discussed in section 3.1.3, the Teukolsky equation separates, yielding the two operators ${}_s\mathcal{R}$ and ${}_s\mathcal{S}$ in equations (3.49a) and (3.49b) (respectively). These operators are analogous to the operators $\mathcal{X}$ and $\mathcal{Y}$ in equation (3.62) above, and depend on derivatives with respect to additional variables t and ϕ. One combination of ${}_s\mathcal{R}$ and ${}_s\mathcal{S}$ is particularly interesting, namely

$${}_s\mathcal{D} \equiv \frac{1}{2}\left({}_s\mathcal{R} - {}_s\mathcal{S}\right). \tag{3.63}$$

One can show that, for $s = 0$, this is in fact the scalar symmetry operator (3.2) discussed by Carter [49].

In the case of linearized gravity, ${}_s\mathcal{D}$ is a map from the space of solutions of the homogeneous Teukolsky equation (3.33) of spin weight s into itself. In section 3.2.4, we will review a procedure (a version of Chrzanowski metric reconstruction [54]) which will allow us to construct another operator ${}_s\mathcal{D}_{ab}{}^{cd}$ from ${}_s\mathcal{D}$ that maps the space of complexified metric perturbations into itself. The symmetry operator ${}_s\mathcal{D}_{ab}{}^{cd}$ will be more useful than ${}_s\mathcal{D}$, since the symplectic product for linearized gravity naturally acts on the space of metric perturbations.

3.2.2 Adjoints and decoupling

In Kerr, for spins higher than 0, there is a second set of irreducible symmetry operators that can be constructed, following an argument due to Wald [177]. This argument holds, as do many of our equations, for all $|s| \leq 2$; however, we will only explicitly use $|s| = 2$ in this chapter.

The argument is as follows. We first define the adjoint of a linear differential operator: consider a linear differential operator $\boldsymbol{\mathcal{L}}$ that takes tensor fields of rank p to tensor fields of rank q. We say that an operator which takes tensor fields of rank q to tensor fields of rank p is the adjoint $\boldsymbol{\mathcal{L}}^\dagger$ of $\boldsymbol{\mathcal{L}}$ if, for all tensor fields $\boldsymbol{\phi}$ of rank p and tensor fields $\boldsymbol{\psi}$ of rank q, there exists a vector field $j^a[\boldsymbol{\phi}, \boldsymbol{\psi}]$ such that

$$\boldsymbol{\psi} \cdot (\boldsymbol{\mathcal{L}} \cdot \boldsymbol{\phi}) - \boldsymbol{\phi} \cdot (\boldsymbol{\mathcal{L}}^\dagger \cdot \boldsymbol{\psi}) = \nabla_a j^a[\boldsymbol{\phi}, \boldsymbol{\psi}]. \tag{3.64}$$

Note that this is not the usual definition of adjoint, which has a complex conjugate acting on $\boldsymbol{\psi}$ in the first term and on $(\mathcal{L}^\dagger \boldsymbol{\psi})$ in the second. Chrzanowski [54] and Gal'tsov [76] use the usual definition, whereas Wald uses the definition (3.64).

We now give some examples of adjoints of the operators considered in section 3.1.3. First, we note that one can easily show that, for two operators $\boldsymbol{\mathcal{L}}_1$ and $\boldsymbol{\mathcal{L}}_2$,

$$(\boldsymbol{\mathcal{L}}_1 \boldsymbol{\mathcal{L}}_2)^\dagger = \boldsymbol{\mathcal{L}}_2^\dagger \boldsymbol{\mathcal{L}}_1^\dagger. \tag{3.65}$$

Moreover, the adjoints of the various Newman-Penrose operators, using equations (3.22), (3.24), and (3.64), are given by

$$D^\dagger = -D - (\epsilon + \bar{\epsilon}) + \rho + \bar{\rho}, \tag{3.66}$$

together with the corresponding expressions obtained via $'$ and $*$ transformations. Using equations (3.64) and (3.16), one finds that ${}_2\boldsymbol{\mathcal{E}}$ is self-adjoint:

$${}_2\boldsymbol{\mathcal{E}}^\dagger = {}_2\boldsymbol{\mathcal{E}}. \tag{3.67}$$

Similarly, one can show from equations (3.66) and (3.34) that

$${}_s\Box^\dagger = {}_{-s}\Box, \tag{3.68}$$

as was first noted by Cohen and Kegeles [56]. Finally, the adjoint of the operator ${}_s\boldsymbol{\tau}$ [equation (3.35)] that enters into the Teukolsky equation (3.33), for $|s| = 2$, is given by

$${}_s\tau^\dagger_{ab} = \begin{cases} [m_{(a|}(D + 2\epsilon - \rho) - l_{(a|}(\delta + 2\beta - \tau)][l_{|b)}(\delta + 4\beta + 3\tau) - m_{|b)}(D + 4\epsilon + 3\rho)] & s = 2 \\ [\bar{m}_{(a|}(\mathbb{\Delta} - 2\gamma + \mu) - n_{(a|}(\bar{\delta} - 2\alpha + \pi)][n_{|b)}(\bar{\delta} - 4\alpha - 3\pi) - \bar{m}_{|b)}(\mathbb{\Delta} - 4\gamma - 3\mu)]\zeta^4 & s = -2 \end{cases}. \tag{3.69}$$

We now take the adjoint of equation (3.36), yielding [from equations (3.68) and (3.67)]

$$ {}_{|s|}\boldsymbol{\mathcal{E}} \cdot {}_{s}\boldsymbol{\tau}^{\dagger} = {}_{s}\boldsymbol{M}^{\dagger}\, {}_{-s}\Box. \tag{3.70}$$

Suppose that we have a solution ${}_{-s}\psi$ to the vacuum Teukolsky equation ${}_{-s}\Box\, {}_{-s}\psi = 0$; note that ${}_{-s}\psi$ is not necessarily the master variable ${}_{-s}\Omega$ associated with δg_{ab} via equation (3.31). Then, from equations (3.70),

$$ {}_{|s|}\boldsymbol{\mathcal{E}} \cdot {}_{s}\boldsymbol{\tau}^{\dagger}\, {}_{-s}\psi = 0. \tag{3.71}$$

Thus, ${}_{s}\boldsymbol{\tau}^{\dagger}\, {}_{-s}\psi$ is a *complex* metric perturbation that solves the vacuum linearized Einstein equations.

The operator ${}_{s}\boldsymbol{\tau}^{\dagger}$ therefore allows the construction of complex vacuum metric perturbations from vacuum solutions to the Teukolsky equation. From a single solution ${}_{-s}\psi$ to the vacuum Teukolsky equation (3.33) of spin weight $-s$, one can apply ${}_{s'}\boldsymbol{M}$ (for some other s', where $|s'| = |s|$) to either ${}_{s}\boldsymbol{\tau}^{\dagger}\, {}_{-s}\psi$ or $\overline{{}_{s}\boldsymbol{\tau}^{\dagger}\, {}_{-s}\psi}$, both of which yield solutions to the vacuum Teukolsky equation:

$$ {}_{s'}\Box\, {}_{s'}\boldsymbol{M} \cdot {}_{s}\boldsymbol{\tau}^{\dagger}\, {}_{-s}\psi = 0, \qquad {}_{s'}\Box\, {}_{s'}\boldsymbol{M} \cdot \overline{{}_{s}\boldsymbol{\tau}^{\dagger}\, {}_{-s}\psi} = 0. \tag{3.72}$$

That is, there exist two symmetry operators of the form

$$ {}_{s',s}\mathcal{C} \equiv {}_{s'}\boldsymbol{M} \cdot {}_{s}\boldsymbol{\tau}^{\dagger}, \qquad {}_{s',s}\widetilde{\mathcal{C}} \equiv {}_{s'}\boldsymbol{M} \cdot \overline{{}_{s}\boldsymbol{\tau}^{\dagger}}. \tag{3.73}$$

The operator ${}_{s',s}\mathcal{C}$ maps from the space of solutions to the vacuum Teukolsky equation (3.33) of spin weight $-s$ to the space of solutions to the vacuum Teukolsky equation of spin weight s'. Similarly, ${}_{s',s}\widetilde{\mathcal{C}}$ maps from the space of solutions to the complex conjugate of the vacuum Teukolsky equation (3.33) of spin weight $-s$ into the space of solutions to the vacuum Teukolsky equation of spin weight s'.

As in section 3.2.1, these operators act on the master variables, rather than metric perturbations. However, one can also construct the operators (for $|s| = 2$)

$$ {}_{s}\mathcal{C}_{ab}{}^{cd} \equiv {}_{s}\tau^{\dagger}_{ab}\, {}_{-s}M^{cd}, \tag{3.74}$$

which are symmetry operators for metric perturbations. That is, they are $\mathbb{R}$-linear maps from the space of complexified solutions to the vacuum linearized Einstein equations into itself. This follows from the operator identity [derived from equations (3.70) and (3.74)]

$$ {}_{|s|}\boldsymbol{\mathcal{E}} \cdot {}_s\boldsymbol{\mathcal{C}} = {}_s\boldsymbol{M}^\dagger\,{}_{-s}\Box\,{}_{-s}\boldsymbol{M} = {}_s\boldsymbol{M}^\dagger\,{}_{-s}\boldsymbol{\tau} \cdot {}_{|s|}\boldsymbol{\mathcal{E}}, \tag{3.75} $$

where the second equality from equation (3.36). Applying this operator identity to (in general) a complex vacuum metric perturbation, the right-hand side yields zero. Note that the two cases $s = \pm 2$ in equations (3.69) and (3.32) differ by a $'$ transformation, along with a factor of ζ^4, and so ${}_2\mathcal{C}_{ab}{}^{cd}$ and ${}_{-2}\mathcal{C}_{ab}{}^{cd}$ are related by a $'$ transformation.

Finally, we note that this argument has been used in a fully tetrad-invariant form, using a spinor form of the Teukolsky equations, to generate symmetry operators for metric perturbations of the sort that we review in this section [10]. For simplicity, we use the Newman-Penrose form of the Teukolsky equations instead.

3.2.3 Issues of gauge

Since the operators ${}_{\pm 2}\tau^\dagger_{ab}$ map into the space of metric perturbations which are solutions to the linearized Einstein equation, the solutions which these operators generate will be in a particular gauge. This gauge freedom can be understood in the following way: the operators ${}_{\pm 2}\tau_{ab}$ in equation (3.33) are only defined up to transformations of the form

$$ {}_{\pm 2}\tau_{ab} \to {}_{\pm 2}\tau_{ab} + 2\xi_{(a}\nabla_{b)}, \tag{3.76} $$

as they act upon the stress-energy tensor, for which $\nabla_a T^{ab} = 0$. As such, we find that ${}_{\pm 2}\tau^\dagger_{ab}$ have the corresponding freedom

$$ {}_{\pm 2}\tau^\dagger_{ab} \to {}_{\pm 2}\tau^\dagger_{ab} + 2\nabla_{(a}\xi_{b)}. \tag{3.77} $$

Note here that, in the second term, the covariant derivative acts upon the argument of these operators in addition to acting on ξ_b. The particular choice (3.35) of ${}_{\pm 2}\tau_{ab}$ fixes this freedom, and so the metric perturbations generated by ${}_{\pm 2}\mathcal{C}_{ab}{}^{cd}$ are in a particular gauge. The gauge conditions which they satisfy are [54]

$$ g^{ab}\,{}_{\pm 2}\tau^\dagger_{ab} = 0, \qquad l^a\,{}_2\tau^\dagger_{ab} = 0, \qquad n^a\,{}_{-2}\tau^\dagger_{ab} = 0. \tag{3.78} $$

For ${}_2\tau^\dagger_{ab}$, this is the *ingoing radiation gauge condition*, whereas for ${}_{-2}\tau^\dagger_{ab}$, this is the *outgoing radiation gauge condition*.

The solutions ${}_2\boldsymbol{\mathcal{C}} \cdot \delta\boldsymbol{g}$ and ${}_{-2}\boldsymbol{\mathcal{C}} \cdot \delta\boldsymbol{g}$, however, do not differ by a gauge transformation. This is in contrast to the case in electromagnetism [82], where the analogous solutions differ by a gauge transformation. While the total solutions ${}_2\boldsymbol{\mathcal{C}}\cdot\delta\boldsymbol{g}$ and ${}_{-2}\boldsymbol{\mathcal{C}}\cdot\delta\boldsymbol{g}$ do not differ by a gauge transformation, we show in the remainder of this section that the imaginary parts of each of these two solutions *are* related by a gauge transformation, and so they represent the same physical solution.

To proceed, we first note the following identities, all derived using equations (3.32), (3.69), and (3.34) [note a conventional factor of two difference with [107], which comes from the difference between their equation (2.21) and our equation (3.19)]

$$\overline{{}_2\boldsymbol{M}} \cdot {}_2\boldsymbol{\mathcal{C}} \mathrel{\hat{=}} \frac{1}{2}(D + \epsilon - 3\bar{\epsilon})(D + 2\epsilon - 2\bar{\epsilon})(D + 3\epsilon - \bar{\epsilon})(D + 4\epsilon)\, {}_{-2}\boldsymbol{M}, \tag{3.79a}$$

$$\overline{{}_{-2}\boldsymbol{M}} \cdot {}_2\boldsymbol{\mathcal{C}} \mathrel{\hat{=}} \frac{1}{2}\bar{\zeta}^4(\delta + 3\bar{\alpha} + \beta)(\delta + 2\bar{\alpha} + 2\beta)(\delta + \bar{\alpha} + 3\beta)(\delta + 4\beta)\, {}_{-2}\boldsymbol{M}, \tag{3.79b}$$

$$\begin{aligned}\overline{{}_{-2}\boldsymbol{M}} \cdot \overline{{}_2\boldsymbol{\mathcal{C}}} &\mathrel{\hat{=}} \frac{3}{2}\overline{\zeta^4\Psi_2}\left[\bar{\tau}(\delta + 4\bar{\alpha}) - \bar{\rho}(\mathbb{\Delta} + 4\bar{\gamma}) - \bar{\mu}(D + 4\bar{\epsilon}) + \bar{\pi}(\bar{\delta} + 4\bar{\beta}) + 2\overline{\Psi}_2\right] \overline{{}_{-2}\boldsymbol{M}} \\ &= \frac{3}{2}\overline{\zeta^3\Psi_2}t^a[\nabla_a + 4(\iota_B\nabla_a o^B)]\overline{{}_{-2}\boldsymbol{M}},\end{aligned} \tag{3.79c}$$

where "$\hat{=}$" means "equality modulo equations of motion". Moreover, apart from those that occur in this equation, all other combinations of ${}_{\pm2}\boldsymbol{M}$ and $\overline{{}_{\pm2}\boldsymbol{M}}$ acting on ${}_2\boldsymbol{\mathcal{C}}$ and $\overline{{}_2\boldsymbol{\mathcal{C}}}$ are zero for vacuum solutions. Here we have used the equation

$$D\rho = (\rho + \epsilon + \bar{\epsilon})\rho \tag{3.80}$$

(along with its $'$- and $*$-transformed versions) in order to simplify, as well as equation (3.39). One can furthermore use a $'$-transformation to write down versions of equation (3.79) involving ${}_{-2}\boldsymbol{\mathcal{C}}$, noting that $\Psi_2 \to \Psi_2$ under a $'$-transformation, and ζ must flip sign (note that t^a keeps the same sign).

To determine whether certain linear combinations of ${}_{\pm2}\mathcal{C}_{ab}{}^{cd}\delta g_{cd}$ (and their complex conjugates) differ by gauge transformations, we need the following relation, which only holds for $\delta\Psi_4$ and $\delta\Psi_0$ coming from the same real vacuum metric perturbations:

$$\begin{aligned}&(D + \epsilon - 3\bar{\epsilon})(D + 2\epsilon - 2\bar{\epsilon})(D + 3\epsilon - \bar{\epsilon})(D + 4\epsilon)\zeta^4\delta\Psi_4 \\ &\quad = (\bar{\delta} - \alpha - 3\bar{\beta})(\bar{\delta} - 2\alpha - 2\bar{\beta})(\bar{\delta} - 3\alpha - \bar{\beta})(\bar{\delta} - 4\alpha)\zeta^4\delta\Psi_0 + 3\overline{\zeta^3\Psi_2}t^a[\nabla_a - 4(\iota_B\nabla_a o^B)]\overline{\delta\Psi_0};\end{aligned} \tag{3.81}$$

we will also need this equation's ′-transform. This relation can be derived using the perturbed Bianchi identities and Newman-Penrose equations, as mentioned in [55]; for a more modern derivation, see for example [9]. Using equations (3.79) and (3.81), along with their ′-transforms, we find that (applied to a real, vacuum metric perturbation),

$$\overline{{}_2\boldsymbol{M}} \cdot {}_2\boldsymbol{\mathcal{C}} \mathrel{\hat{=}} \overline{{}_2\boldsymbol{M}} \cdot {}_{-2}\boldsymbol{\mathcal{C}} - \overline{{}_2\boldsymbol{M}} \cdot \overline{{}_{-2}\boldsymbol{\mathcal{C}}}. \tag{3.82}$$

The ′-transform of this equation merely switches $2 \to -2$. As remarked below equation (3.79), one has that

$$\overline{{}_2\boldsymbol{M}} \cdot \overline{{}_2\boldsymbol{\mathcal{C}}} \mathrel{\hat{=}} 0 \tag{3.83}$$

(along with its ′-transform), and so one therefore has that

$$\overline{{}_2\boldsymbol{M}} \cdot \operatorname{Im}\left[{}_{+2}\boldsymbol{\mathcal{C}} - {}_{-2}\boldsymbol{\mathcal{C}}\right] \cdot \delta\boldsymbol{g} = 0, \qquad \overline{{}_{-2}\boldsymbol{M}} \cdot \operatorname{Im}\left[{}_{+2}\boldsymbol{\mathcal{C}} - {}_{-2}\boldsymbol{\mathcal{C}}\right] \cdot \delta\boldsymbol{g} = 0. \tag{3.84}$$

This equation does not, as it stands, guarantee that $\operatorname{Im}[{}_2\boldsymbol{C} \cdot \delta\boldsymbol{g}]$ and $\operatorname{Im}[{}_{-2}\boldsymbol{C} \cdot \delta\boldsymbol{g}]$ are related by a gauge transformation, just that the master variables associated with these two metric perturbations are equal. This implies that their difference is a metric perturbation that contributes to δM and δa; that is, it only has monopole and dipole terms [175]. One would expect that $\operatorname{Im}[{}_{\pm 2}\mathcal{C}_{ab}{}^{cd}\delta g_{cd}]$, as they are constructed wholly from the radiative Weyl scalars $\delta\Psi_0$ and $\delta\Psi_4$ (which do not have monopole or dipole pieces), would not have non-radiating pieces. This statement is in fact correct due to arguments in [154]. In conclusion, we find that $\operatorname{Im}[{}_2\boldsymbol{C} \cdot \delta\boldsymbol{g}]$ and $\operatorname{Im}[{}_{-2}\boldsymbol{C} \cdot \delta\boldsymbol{g}]$ differ by a gauge transformation:

$$\operatorname{Im}[{}_2\mathcal{C}_{ab}{}^{cd}\delta g_{cd}] = \operatorname{Im}[{}_{-2}\mathcal{C}_{ab}{}^{cd}\delta g_{cd}] + 2\nabla_{(a}\xi_{b)}, \tag{3.85}$$

for some vector field ξ^a. The main theorem of [9] provides an alternative proof of this result, as does the discussion in section 4.3 of [10].

3.2.4 Diagonal action on expansions

In section 3.1.3, we showed that the master variables (and their complex conjugates) have convenient expansions [equations (3.58) and (3.59)] in terms of spin-weighted spheroidal harmonics. We show in this section that the symmetry operators considered in this chapter which act on the master variables are "diagonal", in the sense that they act upon each term in these expansions by simply multiplying each term by an overall constant. We then construct a similar expansion for vacuum metric perturbations, and show that the action of the symmetry operators that we have defined for metric perturbations are also diagonal on this expansion.

3.2.4.1 Symmetry operators for the master variables and the Teukolsky-Starobinsky identities

First, let us consider the action of the symmetry operator ${}_s\mathcal{D}$ defined in equation (3.63). From equations (3.49), (3.43), (3.45), and (3.51), it follows that

$$ {}_s\mathcal{D}\,{}_s\Omega = \int_{-\infty}^{\infty}\sum_{l=2}^{\infty}\sum_{|m|\leq l}\sum_{p=\pm 1}{}_{|s|}\lambda_{lm\omega}e^{i(m\phi-\omega t)}\,{}_s\Theta_{lm\omega}(\theta)\,{}_s\widehat{\Omega}_{lm\omega p}(r). \tag{3.86}$$

In section 3.2.4.3, we will also show that a similar diagonalization occurs for a tensor version of this operator, which we will define in equation (3.107).

Next, we consider the symmetry operators ${}_{s',s}\widetilde{\mathcal{C}}$ defined in equation (3.73). We begin by noting that these symmetry operators simplify with the choice of Boyer-Lindquist coordinates and the Kinnersley tetrad, yielding the so-called "spin-inversion" operators [54, 76]:

$$ {}_{2,2}\widetilde{\mathcal{C}} = \frac{1}{2}\mathscr{D}_0^4, \qquad {}_{-2,-2}\widetilde{\mathcal{C}} = \frac{1}{32}\Delta^2\left(\mathscr{D}_0^{+}\right)^4\Delta^2, \tag{3.87a}$$

$$ {}_{2,-2}\widetilde{\mathcal{C}} = \frac{1}{8}\mathscr{L}_{-1}^{+}\mathscr{L}_0^{+}\mathscr{L}_1^{+}\mathscr{L}_2^{+}, \qquad {}_{-2,2}\widetilde{\mathcal{C}} = \frac{1}{8}\mathscr{L}_{-1}\mathscr{L}_0\mathscr{L}_1\mathscr{L}_2. \tag{3.87b}$$

The constant numerical factors here are consistent with those of Wald [177] and Chrzanowski [54], but disagree with those of other authors (such as [52, 76]) due to normalization conventions.

These operators are referred to as spin-inversion operators for the following reason. Considering their action on the terms in the expansion (3.59) of $\overline{{}_s\Omega}$, they are either purely radial [equation (3.87a)] or purely angular [equation (3.87b)]. Due to this fact, along with the expansions in equations (3.58) and (3.59), it is apparent that, when acting on the terms in these expansions, the operator ${}_{2,2}\widetilde{\mathcal{C}}$ maps from the space of solutions to the radial Teukolsky equation (3.51b) with $s=-2$ to $s=2$, and similarly ${}_{-2,-2}\widetilde{\mathcal{C}}$ maps from solutions with $s=2$ to $s=-2$. Similarly, for the angular operators, due to the fact that the expansion for $\overline{{}_s\Omega}$ is in terms of ${}_{-s}\Theta_{lm\omega}$, ${}_{2,-2}\widetilde{\mathcal{C}}$ maps from the space of solutions to angular Teukolsky equation (3.51a) with $s=2$ to $s=-2$, and similarly ${}_{-2,2}\widetilde{\mathcal{C}}$ maps from $s=-2$ to $s=2$.

We now show that the spin-inversion operators merely multiply each term in the expansion (3.59) by some constant, starting with the angular spin-inversion operators. The angular Teukolsky equation (3.51a) is a Sturm-Liouville problem, and so it only has one solution for a given value of l, m, and ω (up to normalization). If the angular spin-inversion operators, when acting upon individual terms in the expansion (3.59), map between the two spaces of solutions with $s = \pm 2$, then these maps can be entirely characterized by two overall constants, which we denote by ${}_{\pm 2}C_{lm\omega}$:

$$\mathscr{L}_{-1(\pm m)(\pm\omega)}\mathscr{L}_{0(\pm m)(\pm\omega)}\mathscr{L}_{1(\pm m)(\pm\omega)}\mathscr{L}_{2(\pm m)(\pm\omega)}\,{}_{\pm 2}\Theta_{lm\omega} \equiv {}_{\pm 2}C_{lm\omega}\,{}_{\mp 2}\Theta_{lm\omega}. \tag{3.88}$$

This equation is known as the *angular Teukolsky-Starobinsky identity*. Since these operators are entirely real, this constant ${}_{\pm 2}C_{lm\omega}$ is also real. Moreover, the normalization condition for ${}_s\Theta_{lm\omega}$ implies that [52]

$${}_2C_{lm\omega} = {}_{-2}C_{lm\omega} \equiv C_{lm\omega}, \tag{3.89}$$

where

$$C^2_{lm\omega} = {}_2\lambda^2_{lm\omega}({}_2\lambda_{lm\omega}+2)^2 - 8\omega^2\,{}_2\lambda_{lm\omega}[\alpha^2_{m\omega}(5\,{}_2\lambda_{lm\omega}+6) - 12a^2] + 144\omega^4\alpha^4_{m\omega}, \tag{3.90}$$

and

$$\alpha^2_{m\omega} = a^2 - am/\omega. \tag{3.91}$$

We now turn to the case of the radial operators in equation (3.87a), which are somewhat more complicated. This is because there are two solutions to the radial equation (3.51b), as it is second-order, and not a Sturm-Liouville problem. However, as noted in section 3.1.3, the two solutions can be characterized by their eigenvalues p under the transformation $(m,\omega) \to (-m,-\omega)$, followed by complex conjugation. Since the radial spin-inversion operator is also invariant under this transformation, we must therefore have that

$$\Delta^2\mathscr{D}^4_{0(\mp m)(\mp\omega)}\Delta^{(s\pm s)/2}\,{}_{\pm 2}\widehat{\Omega}_{lm\omega p} \equiv 2^{\pm 2}\,{}_{\pm 2}C_{lm\omega p}\Delta^{(s\mp s)/2}\,{}_{\mp 2}\widehat{\Omega}_{lm\omega p} \tag{3.92}$$

(the factor of $2^{\pm 2}$ is purely conventional, and is present only to make our final expressions simpler). This equation is known as the *radial Teukolsky-Starobinsky identity*.

To determine the values of the constants ${}_{\pm 2}C_{lm\omega p}$, we need to use the fact that ${}_{\pm 2}\Omega$ come from the same real metric perturbation. The values of these constants given by Teukolsky and Press in their original paper [163] only hold for the $p = 1$ case (as pointed out by Bardeen [27][6]). The values of ${}_{\pm 2}C_{lm\omega p}$ are found using equation (3.82), since (in terms of ${}_s\Omega$) the complex conjugate of this equation (and its ′-transform) can be written as

$$ {}_{-s,-s}\widetilde{\mathcal{C}}\;\overline{{}_s\Omega} = {}_{-s,s}\widetilde{\mathcal{C}}\;\overline{{}_{-s}\Omega} - {}_{-s,s}\mathcal{C}\;{}_{-s}\Omega. \tag{3.93} $$

Using equations (3.87), (3.88), and (3.92), as well as (3.79c), we find that

$$ {}_{-s,s}\widetilde{\mathcal{C}}\;\overline{{}_{-s}\Omega} = \frac{1}{8}\int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=2}^{\infty}\sum_{|m|\leq l}\sum_{p=\pm 1} pC_{lm\omega}e^{i(m\phi-\omega t)}\;{}_{-s}\Theta_{lm\omega}\;{}_{-s}\widehat{\Omega}_{lm\omega p}, \tag{3.94a} $$

$$ {}_{-s,-s}\widetilde{\mathcal{C}}\;\overline{{}_{s}\Omega} = \frac{1}{8}\int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=2}^{\infty}\sum_{|m|\leq l}\sum_{p=\pm 1} p\;{}_sC_{lm\omega p}e^{i(m\phi-\omega t)}\;{}_{-s}\Theta_{lm\omega}\;{}_{-s}\widehat{\Omega}_{lm\omega p}, \tag{3.94b} $$

$$ {}_{-s,s}\mathcal{C}\;{}_{-s}\Omega = \frac{3iM}{2}\,\mathrm{sgn}(s)\int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=2}^{\infty}\sum_{|m|\leq l}\sum_{p=\pm 1} \omega e^{i(m\phi-\omega t)}\;{}_{-s}\Theta_{lm\omega}\;{}_{-s}\widehat{\Omega}_{lm\omega p}, \tag{3.94c} $$

and so equation (3.93) implies that

$$ {}_{\pm 2}C_{lm\omega p} = C_{lm\omega} \mp 12ipM\omega. \tag{3.95} $$

3.2.4.2 Debye potentials and an expansion for the metric perturbation

At this point, we have shown how symmetry operators on the space of master variables act diagonally on the expansions (3.58) and (3.59). We would like a similar diagonalization for the operator ${}_s\boldsymbol{\mathcal{C}}$, but (*a priori*) there does not exist an analogous expansion for the metric perturbation. We now construct such an expansion. To begin, if ${}_s\psi$ is the master variable associated with some real solution (not δg_{ab}) of the vacuum linearized Einstein equations and if

$$ {}_s\Omega = {}_sM^{ab}\,\mathrm{Im}[\,{}_s\tau^\dagger_{ab}\;{}_{-s}\psi], \tag{3.96} $$

then we call ${}_s\psi$ a *Debye potential* for δg_{ab} (for the origin of this terminology, see [56]). The first of these conditions ensures that ${}_2\psi$ and $\zeta^{-4}\;{}_{-2}\psi$ satisfy the same relation as (respectively) $\delta\Psi_0$ and

[6]That [163] only considers $p = 1$ can be seen from their equation 3.21, along with the remark below their equation 3.22 that the quantities S_2 and $S_2^\dagger$ that appear in this equation are given by ${}_2S_{lm}$ and ${}_{-2}S_{lm}$ (in this chapter, these are denoted ${}_2\Theta_{lm\omega}$ and ${}_{-2}\Theta_{lm\omega}$). These two statements imply that the radial functions R_s discussed in [163] obey

$$ \overline{R_s(-m,-\omega)} = R_s(m,\omega). $$

In this chapter, due to differences in notation and the conventions in equation (3.52), this is equivalent to the statement that ${}_{\pm 2}\widehat{\Omega}_{l(-m)(-\omega)} = (-1)^{m+s}\,\overline{{}_s\widehat{\Omega}_{lm\omega}}$, which by equation (3.60) implies that $p = 1$.

$\delta\Psi_4$ in equation (3.81). The second of these conditions ensures that $\mathrm{Im}[{}_s\tau^{\dagger}_{ab}\,{}_{-s}\psi]$ and (by the first condition) $\mathrm{Im}[{}_{-s}\tau^{\dagger}_{ab}\,{}_s\psi]$ are the same as δg_{ab}, up to gauge and $l = 0, 1$ terms.

The easiest way to satisfy these conditions is as follows. First, note that, by equations (3.74) and (3.94),

$$\begin{aligned} &{}_sM^{ab}\,\mathrm{Im}\left\{{}_s\mathcal{C}_{ab}{}^{cd}\,\mathrm{Im}[{}_{-s}\tau^{\dagger}_{cd}\,{}_s\Omega]\right\} \\ &\quad= \frac{1}{16}\,{}_sM^{ab}\,\mathrm{Re}\left[{}_s\tau^{\dagger}_{ab}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\le l}\sum_{p=\pm1} p\,{}_sC_{lm\omega p}e^{i(m\phi-\omega t)}\,{}_{-s}\Theta_{lm\omega}\,{}_{-s}\widehat{\Omega}_{lm\omega p}\right] \\ &\quad= \frac{1}{256}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\le l}\sum_{p=\pm1}(C^2_{lm\omega}+144M^2\omega^2)e^{i(m\phi-\omega t)}\,{}_s\Theta_{lm\omega}\,{}_s\widehat{\Omega}_{lm\omega p}. \end{aligned} \tag{3.97}$$

We now define ${}_s\psi$, for a given ${}_s\Omega$, by

$$\begin{aligned} {}_s\psi &\equiv 256\,{}_sM^{ab}\,\mathrm{Im}\left[{}_s\tau^{\dagger}_{ab}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\le l}\sum_{p=\pm1}\frac{e^{i(m\phi-\omega t)}\,{}_{-s}\Theta_{lm\omega}(\theta)\,{}_{-s}\widehat{\Omega}_{lm\omega p}(r)}{C^2_{lm\omega}+144M^2\omega^2}\right] \\ &= 16i\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\le l}\sum_{p=\pm1}\frac{pe^{i(m\phi-\omega t)}\,{}_s\Theta_{lm\omega}(\theta)\,{}_s\widehat{\Omega}_{lm\omega p}(r)}{{}_sC_{lm\omega p}}, \end{aligned} \tag{3.98}$$

where the second line comes from equation (3.94), and ${}_s\widehat{\Omega}_{lm\omega p}$ is given in terms of ${}_s\Omega$ by equations (3.50) and (3.56). Since $C^2_{lm\omega}+144M^2\omega^2$ is real, ${}_s\psi$ satisfies the first of the above requirements, and by equation (3.97) it also satisfies the second. Moreover, the second line implies that

$${}_s\widehat{\psi}_{lm\omega(-p)} = \frac{16ip}{{}_sC_{lm\omega p}}\,{}_s\widehat{\Omega}_{lm\omega p}. \tag{3.99}$$

where the expansion coefficients ${}_s\widehat{\psi}_{lm\omega p}$ are defined by an expansion analogous to equation (3.58), together with the behavior under complex conjugation given by equation (3.60). This condition is satisfied, due to the fact that

$$\overline{{}_sC_{l(-m)(-\omega)p}} = {}_sC_{lm\omega p}, \tag{3.100}$$

by equations (3.53), (3.90) and (3.95), as well as by using equation (3.60) for ${}_s\widehat{\Omega}_{lm\omega p}$. While this would also be a perfectly reasonable definition of ${}_s\psi$, it is not apparent in this form that ${}_s\psi$ is generated by a real metric perturbation, which is crucial, and is explicit in equation (3.98). Finally, note that equations analogous to equation (3.94) also hold for ${}_s\psi$ in terms of ${}_s\psi_{lm\omega p}$.

We can now define an expansion for the metric perturbation. First, we define

$$\delta_{\pm}g_{ab} \equiv {}_{\pm2}\tau^{\dagger}_{ab}\,{}_{\mp2}\psi, \tag{3.101}$$

which (as remarked above) satisfy

$$ {}_sM^{ab}\,\mathrm{Im}[\eth_+ g_{ab}] = {}_sM^{ab}\,\mathrm{Im}[\eth_- g_{ab}] = {}_s\Omega. \tag{3.102}$$

These metric perturbations have convenient expansions of the form

$$\eth_\pm g_{ab} = \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=2}^{\infty} \sum_{|m|\le l} \sum_{p=\pm 1} (\eth_\pm g_{lm\omega p})_{ab}, \tag{3.103}$$

where

$$(\eth_\pm g_{lm\omega p})_{ab} \equiv {}_{\pm 2}\tau^\dagger_{ab}\left[e^{i(m\phi-\omega t)}\,{}_{\mp 2}\Theta_{lm\omega}(\theta)\,{}_{\mp 2}\widehat{\psi}_{lm\omega p}(r)\right]. \tag{3.104}$$

Note that the relationship between $\eth_\pm g_{ab}$ and their coefficients is not $\mathbb{C}$-linear, due to the transformation properties of these coefficients under complex conjugation resulting from equation (3.60).

This procedure, which allowed us to construct a metric perturbation $\mathrm{Im}[\eth_\pm g_{ab}]$ from ${}_{\mp 2}\Omega$ such that the master variables associated with this metric perturbation are ${}_{\pm 2}\Omega$, is similar to the one laid out in [54], which is referred to in the literature as *Chrzanowski metric reconstruction*. We now provide an operator form of this procedure: define

$$\begin{aligned} {}_s\Pi_{ab}\,{}_s\Omega &\equiv 256\,{}_s\mathcal{C}_{ab}{}^{cd}\,\mathrm{Im}\left[-{}_s\tau^\dagger_{cd}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\le l}\sum_{p=\pm1}\frac{e^{i(m\phi-\omega t)}\,{}_s\Theta_{lm\omega}\,{}_s\widehat{\Omega}_{lm\omega p}}{C^2_{lm\omega}+144M^2\omega^2}\right] \\ &= 16i\,{}_s\tau^\dagger_{ab}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\le l}\sum_{p=\pm1}\frac{pe^{i(m\phi-\omega t)}\,{}_{-s}\Theta_{lm\omega}\,{}_{-s}\widehat{\Omega}_{lm\omega p}}{{}_{-s}C_{lm\omega p}}, \end{aligned} \tag{3.105}$$

which satisfies

$$ {}_sM^{ab}\,\mathrm{Im}[{}_s\Pi_{ab}\,{}_s\Omega] = {}_sM^{ab}\,\mathrm{Im}[{}_{-s}\Pi_{ab}\,{}_{-s}\Omega] = {}_s\Omega. \tag{3.106}$$

Note that the operator ${}_s\Pi_{ab}$ is non-local, since it requires an expansion in spin-weighted spheroidal harmonics for its definition. This operator allows us to define a version of the operator ${}_s\mathcal{D}$ defined in section 3.2.1 that maps to the space of complexified solutions of the linearized Einstein equations, much like ${}_s\mathcal{C}_{ab}{}^{cd}$:

$$ {}_s\mathcal{D}_{ab}{}^{cd} \equiv {}_s\Pi_{ab}\,{}_s\mathcal{D}\,{}_sM^{cd}. \tag{3.107}$$

We also define a version of this operator *without* the intermediate factor of ${}_s\mathcal{D}$:

$$ {}_s\mathrm{X}_{ab}{}^{cd} \equiv {}_s\Pi_{ab}\,{}_sM^{cd}. \tag{3.108}$$

3.2.4.3 Diagonal action of our operators on the metric perturbation

Now that we have both a definition of an expansion for the metric perturbation, along with a variety of symmetry operators defined which map the space of metric perturbations into itself, we can proceed to show that these symmetry operators act diagonally on these expansions. Note, again, that there is no convenient notion of an expansion of the form (3.103) for a general $\eth g_{ab}$, and so we only compute the action of our various symmetry operators on $\eth_\pm g_{ab}$. The simplest case is ${}_s\mathcal{C}_{ab}{}^{cd}$, which satisfies [by equation (3.94)][7]

$$\begin{aligned} {}_{\pm 2}\mathcal{C}_{ab}{}^{cd}\overline{\eth_\pm g_{cd}} &= {}_{\pm 2}\tau^\dagger_{ab}\,{}_{\mp 2,\pm 2}\widetilde{\mathcal{C}}\;\overline{{}_{\mp 2}\psi} \\ &= \frac{1}{8}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\leq l}\sum_{p=\pm 1} pC_{lm\omega}(\eth_\pm g_{lm\omega p})_{ab}, \end{aligned} \tag{3.109a}$$

$$\begin{aligned} {}_{\pm 2}\mathcal{C}_{ab}{}^{cd}\overline{\eth_\mp g_{cd}} &= {}_{\pm 2}\tau^\dagger_{ab}\,{}_{\mp 2,\mp 2}\widetilde{\mathcal{C}}\;\overline{{}_{\pm 2}\psi} \\ &= \frac{1}{8}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\leq l}\sum_{p=\pm 1} p\,{}_{\pm 2}C_{lm\omega p}(\eth_\pm g_{lm\omega p})_{ab}, \end{aligned} \tag{3.109b}$$

$$\begin{aligned} {}_{\pm 2}\mathcal{C}_{ab}{}^{cd}\eth_\pm g_{cd} &= {}_{\pm 2}\tau^\dagger_{ab}\,{}_{\mp 2,\pm 2}\mathcal{C}\,{}_{\mp 2}\psi \\ &= \pm\frac{3iM}{2}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\leq l}\sum_{p=\pm 1} \omega(\eth_\pm g_{lm\omega p})_{ab}. \end{aligned} \tag{3.109c}$$

These equations demonstrate that the action on the expansion (3.103) is diagonal, up to mappings from $\overline{(\eth_\pm g_{lm\omega p})_{ab}} \to (\eth_\pm g_{lm\omega p})_{ab}$ and $(\eth_\mp g_{lm\omega p})_{ab}$, as well as mappings from $(\eth_\pm g_{lm\omega p})_{ab} \to (\eth_\mp g_{lm\omega p})_{ab}$. More useful later in this chapter will be the action of ${}_s\mathcal{C}_{ab}{}^{cd}$ on $\mathrm{Im}[\eth_\pm g_{ab}]$:

$$\begin{aligned} {}_{\pm 2}\mathcal{C}_{ab}{}^{cd}\,\mathrm{Im}[\eth_+ g_{cd}] &= {}_{\pm 2}\mathcal{C}_{ab}{}^{cd}\,\mathrm{Im}[\eth_- g_{cd}] \\ &= \frac{i}{16}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l=2}^{\infty}\sum_{|m|\leq l}\sum_{p=\pm 1} p\,{}_{\pm 2}C_{lm\omega p}(\eth_\pm g_{lm\omega p})_{ab}. \end{aligned} \tag{3.110}$$

Similarly, we will consider the action of ${}_s\mathcal{D}_{ab}{}^{cd}$ and ${}_s\mathrm{X}_{ab}{}^{cd}$ on $\mathrm{Im}[\eth_\pm g_{ab}]$. We have that [by equation (3.102)]

$${}_s\Pi_{ab}\,{}_s\Omega = {}_s\mathrm{X}_{ab}{}^{cd}\,\mathrm{Im}[\eth_\pm g_{cd}], \tag{3.111}$$

[7] Note that, as mentioned above below equation (3.104), the relationship between $\eth_\pm g_{ab}$ and their coefficients is not $\mathbb{C}$-linear. This explains the apparent contradiction of the left-hand side of equations (3.109a) and (3.109b) being $\mathbb{C}$-antilinear, but the right-hand sides appearing to be $\mathbb{C}$-linear.

along with [by equations (3.101) and (3.105)]

$$ {}_{\pm 2}\Pi_{ab}\,{}_{\pm 2}\Omega = \eth_{\pm} g_{ab}, \tag{3.112} $$

and so we find that

$$ {}_{\pm 2}\mathrm{X}_{ab}{}^{cd}\,\mathrm{Im}[\eth_{+} g_{cd}] = {}_{\pm 2}\mathrm{X}_{ab}{}^{cd}\,\mathrm{Im}[\eth_{-} g_{cd}] = \eth_{\pm} g_{ab}, \tag{3.113} $$

Similarly, by the $\mathbb{R}$-linearity of equation (3.112), we find that [from equation (3.86)]

$$ {}_{\pm 2}\mathcal{D}_{ab}{}^{cd}\,\mathrm{Im}[\eth_{+} g_{cd}] = {}_{\pm 2}\mathcal{D}_{ab}{}^{cd}\,\mathrm{Im}[\eth_{-} g_{cd}] = \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=2}^{\infty} \sum_{|m|\leq l} \sum_{p=\pm 1} {}_{2}\lambda_{lm\omega} (\eth_{\pm} g_{lm\omega p})_{ab}. \tag{3.114} $$

3.2.5 Projection operators

The final set of symmetry operators that we introduce are projection operators acting on the space of master variables ${}_s\Omega$. Before we introduce these operators, however, it is relevant to discuss the asymptotic properties of the master variables. First, define the tortoise coordinate r^* by

$$ \frac{\mathrm{d}r^*}{\mathrm{d}r} \equiv \frac{r^2 + a^2}{\Delta}. \tag{3.115} $$

This coordinate satisfies $r^* \to \infty$ as $r \to \infty$ and $r^* \to -\infty$ as $r \to r_+$, where r_+ is the location of the horizon, satisfying $\Delta|_{r=r_+} = 0$.

Now, the vacuum Teukolsky radial equation (3.51b) is a second-order ordinary differential equation in r, and so its solution space is spanned by two solutions (for given values of s, l, m, and ω) that are characterized by their asymptotic behavior at either $r = r_+$ or $r = \infty$. One can show, from the asymptotic form of the vacuum Teukolsky radial equation (3.51b), that one can choose two independent solutions ${}_sR^{\mathrm{in}}_{lm\omega}(r)$ and ${}_sR^{\mathrm{out}}_{lm\omega}(r)$ with the following asymptotic forms as $r^* \to -\infty$ [163]:

$$ {}_sR^{\mathrm{in}}_{lm\omega}(r) \to e^{-ik_{m\omega} r^*}/\Delta^s, \quad {}_sR^{\mathrm{out}}_{lm\omega}(r) \to e^{ik_{m\omega} r^*}, \tag{3.116} $$

where

$$ k_{m\omega} \equiv \omega - am/(2Mr_+). \tag{3.117} $$

Similarly, at $r^* \to \infty$, one can choose two independent solutions ${}_sR^{\mathrm{down}}_{lm\omega}(r)$ and ${}_sR^{\mathrm{up}}_{lm\omega}(r)$, which have the following asymptotic forms:

$$ {}_sR^{\mathrm{down}}_{lm\omega}(r) \to e^{-i\omega r^*}/r, \quad {}_sR^{\mathrm{up}}_{lm\omega}(r) \to e^{i\omega r^*}/r^{2s+1}. \tag{3.118} $$

A general solution can therefore be expanded in terms of these solutions as

$$\begin{aligned} {}_s\widehat{\Omega}_{lm\omega}(r) &= {}_s\widehat{\Omega}^{\text{down}}_{lm\omega}\, {}_sR^{\text{down}}_{lm\omega}(r) + {}_s\widehat{\Omega}^{\text{up}}_{lm\omega}\, {}_sR^{\text{up}}_{lm\omega}(r) \\ &= {}_s\widehat{\Omega}^{\text{in}}_{lm\omega}\, {}_sR^{\text{in}}_{lm\omega}(r) + {}_s\widehat{\Omega}^{\text{out}}_{lm\omega}\, {}_sR^{\text{out}}_{lm\omega}(r). \end{aligned} \tag{3.119}$$

Moreover, from the asymptotic behavior in equations (3.116) and (3.118), we have

$$\overline{{}_sR^{\text{in/out/down/up}}_{l(-m)(-\omega)}(r)} = {}_sR^{\text{in/out/down/up}}_{lm\omega}(r), \tag{3.120}$$

and so, from the definition (3.56),

$$\begin{aligned} {}_s\widehat{\Omega}_{lm\omega p}(r) &= {}_s\widehat{\Omega}^{\text{down}}_{lm\omega p} R^{\text{down}}_{lm\omega}(r) + {}_s\widehat{\Omega}^{\text{up}}_{lm\omega p} R^{\text{up}}_{lm\omega}(r) \\ &= {}_s\widehat{\Omega}^{\text{in}}_{lm\omega p} R^{\text{in}}_{lm\omega}(r) + {}_s\widehat{\Omega}^{\text{out}}_{lm\omega p} R^{\text{out}}_{lm\omega}(r), \end{aligned} \tag{3.121}$$

where

$${}_s\widehat{\Omega}^{\text{in/out/down/up}}_{lm\omega p} \equiv \frac{1}{2}\left[{}_s\widehat{\Omega}^{\text{in/out/down/up}}_{lm\omega} + p(-1)^{m+s}\, \overline{{}_s\widehat{\Omega}^{\text{in/out/down/up}}_{l(-m)(-\omega)}} \right]. \tag{3.122}$$

One minor note that will be useful later in this chapter is that the coefficients ${}_s\widehat{\Omega}^{\text{in/out/down/up}}_{lm\omega p}$ and ${}_{-s}\widehat{\Omega}^{\text{in/out/down/up}}_{lm\omega p}$ are not independent. To show this, consider the following expressions for the leading order form of the derivative operators $\mathscr{D}_{0(\pm m)(\pm\omega)}$ acting on exponentials of the form $e^{\pm i\omega r^*}$ or $e^{\mp i\omega r^*}$:

$$\left.\begin{aligned} \mathscr{D}_{0(\pm m)(\pm\omega)} f(r) e^{\pm i\omega r^*} &= \frac{\mathrm{d}f}{\mathrm{d}r} e^{\pm i\omega r^*} \\ \mathscr{D}_{0(\pm m)(\pm\omega)} f(r) e^{\mp i\omega r^*} &= \left[\frac{\mathrm{d}f}{\mathrm{d}r} \mp 2i\omega f(r)\right] e^{\mp i\omega r^*} \end{aligned}\right\} r^* \to \infty, \tag{3.123a}$$

$$\left.\begin{aligned} \mathscr{D}_{0(\pm m)(\pm\omega)} f(r) e^{\pm ik_{m\omega} r^*} &= \frac{\mathrm{d}f}{\mathrm{d}r} e^{\pm ik_{m\omega} r^*} \\ \mathscr{D}_{0(\pm m)(\pm\omega)} f(r) e^{\mp ik_{m\omega} r^*} &= \left[\frac{\mathrm{d}f}{\mathrm{d}r} \mp \frac{4Mr^+}{\Delta} ik_{m\omega} f(r)\right] e^{\mp ik_{m\omega} r^*} \end{aligned}\right\} r^* \to -\infty. \tag{3.123b}$$

Using these equations, the radial Teukolsky-Starobinsky identity (3.92), and the asymptotic forms (3.116) and (3.118), one can show that

$${}_2\widehat{\Omega}^{\text{in}}_{lm\omega p} = \frac{(4Mr_+k_{m\omega})^4\, {}_1\kappa_{m\omega}\, {}_0\kappa_{m\omega}\, {}_{-1}\kappa_{m\omega}\, {}_{-2}\kappa_{m\omega}}{{}_{-2}C_{lm\omega p}/4}\, {}_{-2}\widehat{\Omega}^{\text{in}}_{lm\omega p}, \tag{3.124a}$$

$${}_{-2}\widehat{\Omega}^{\text{out}}_{lm\omega p} = \frac{(4Mr_+k_{m\omega})^4\, {}_{-1}\kappa_{m\omega}\, {}_0\kappa_{m\omega}\, {}_1\kappa_{m\omega}\, {}_2\kappa_{m\omega}}{4\, {}_2C_{lm\omega p}}\, {}_2\widehat{\Omega}^{\text{out}}_{lm\omega p}, \tag{3.124b}$$

$${}_2\widehat{\Omega}^{\text{down}}_{lm\omega p} = \frac{(2\omega)^4}{C_{lm\omega p}/4}\, {}_{-2}\widehat{\Omega}^{\text{down}}_{lm\omega p}, \tag{3.124c}$$

$${}_{-2}\widehat{\Omega}^{\text{up}}_{lm\omega p} = \frac{(2\omega)^4}{4C_{lm\omega p}}\, {}_2\widehat{\Omega}^{\text{up}}_{lm\omega p}, \tag{3.124d}$$

where

$$ {}_s\kappa_{m\omega} = 1 - \frac{is(r_+ - M)}{2Mr_+ k_{m\omega}}. \tag{3.125} $$

We now define projection operators associated with this expansion as follows: for example, define ${}_s\mathcal{P}^{\text{in}}$ by

$$ \begin{aligned} {}_s\mathcal{P}^{\text{in}}\, {}_s\Omega = {}_s\mathcal{P}^{\text{in}} \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=|s|}^{\infty} \sum_{|m|\leq l} e^{i(m\phi-\omega t)}\, {}_s\Theta_{lm\omega}(\theta) \left[{}_s\widehat{\Omega}^{\text{in}}_{lm\omega}\, {}_sR^{\text{in}}_{lm\omega}(r) + {}_s\widehat{\Omega}^{\text{out}}_{lm\omega}\, {}_sR^{\text{out}}_{lm\omega}(r) \right] \\ \equiv \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=|s|}^{\infty} \sum_{|m|\leq l} e^{i(m\phi-\omega t)}\, {}_s\Theta_{lm\omega}(\theta)\, {}_s\widehat{\Omega}^{\text{in}}_{lm\omega}\, {}_sR^{\text{in}}_{lm\omega}(r). \end{aligned} \tag{3.126} $$

Analogous definitions can be given for ${}_s\mathcal{P}^{\text{out}}$, ${}_s\mathcal{P}^{\text{down}}$, and ${}_s\mathcal{P}^{\text{up}}$. Since these operators require an expansion in spin-weighted spheroidal harmonics, they are necessarily non-local.

The reason we introduce these projection operators is that, as we show in section 3.5.2, whether ${}_s\tau^\dagger_{ab}\, {}_{-s}\Omega$ falls off as $1/r$ (that is, whether it is an asymptotically flat metric perturbation) depends on the values ${}_{-s}\Omega^{\text{down/out}}_{lm\omega}$. This was first remarked by Chrzanowski in [54]. As such, we define a projected version of ${}_s\tau^\dagger_{ab}$, which we call ${}_s\mathring{\tau}^\dagger_{ab}$, such that ${}_s\mathring{\tau}^\dagger_{ab}\, {}_{-s}\Omega$ is always well-behaved as $r \to \infty$:

$$ {}_2\mathring{\tau}^\dagger_{ab} \equiv {}_2\tau^\dagger_{ab}\, {}_{-2}\mathcal{P}^{\text{down}}, \qquad {}_{-2}\mathring{\tau}^\dagger_{ab} \equiv {}_{-2}\tau^\dagger_{ab}\, {}_2\mathcal{P}^{\text{up}}. \tag{3.127} $$

Using this operator, we can define

$$ {}_s\mathring{\mathcal{C}}_{ab}{}^{cd} \equiv {}_s\mathring{\tau}^\dagger_{ab}\, {}_{-s}M^{cd}, \tag{3.128} $$

which allows for the definition of

$$ {}_s\mathring{\Pi}_{ab}{}^{cd}\, {}_s\Omega \equiv 256\, {}_s\mathring{\mathcal{C}}_{ab}{}^{cd}\, \mathrm{Im} \left[{}_{-s}\tau^\dagger_{cd} \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{l=2}^{\infty} \sum_{|m|\leq l} \sum_{p=\pm 1} \frac{e^{i(m\phi-\omega t)}\, {}_s\Theta_{lm\omega}\, {}_s\widehat{\Omega}_{lm\omega p}}{C^2_{lm\omega} + 144M^2\omega^2} \right]. \tag{3.129} $$

Finally, this last operator allows for the definitions

$$ {}_s\mathring{\mathcal{D}}_{ab}{}^{cd} \equiv {}_s\mathring{\Pi}_{ab}\, {}_s\mathcal{D}\, {}_sM^{cd}, \qquad {}_s\mathring{\mathrm{X}}_{ab}{}^{cd} \equiv {}_s\mathring{\Pi}_{ab}\, {}_sM^{cd}. \tag{3.130} $$

3.3 Conserved Currents

We next turn to conserved currents that can be constructed using these symmetry operators. First, we review the general theory of symplectic products, which are bilinear currents constructed from the Lagrangian formulation of a given classical field theory. We then select a handful of conserved currents that can be constructed using symplectic products and symmetry operators, whose properties we discuss throughout the rest of this chapter.

3.3.1 Symplectic currents

Given a theory which possesses a Lagrangian formulation, one method of generating conserved quantities is to use the symplectic product defined in this section. Following Burnett and Wald [44], starting from a Lagrangian density $\mathcal{L}[\boldsymbol{\phi}]$ that is locally constructed from dynamical fields $\boldsymbol{\phi}$, we consider a Lagrangian four-form $\boldsymbol{L}[\boldsymbol{\phi}] \equiv {}^*\mathcal{L}[\boldsymbol{\phi}]$, where * denotes the Hodge dual. It then follows that

$$\delta \boldsymbol{L}[\boldsymbol{\phi}] \equiv \boldsymbol{E}[\boldsymbol{\phi}] \cdot \delta\boldsymbol{\phi} - \mathrm{d}\boldsymbol{\theta}[\boldsymbol{\phi}; \delta\boldsymbol{\phi}], \tag{3.131}$$

where the three-form $\boldsymbol{\theta}[\boldsymbol{\phi}; \delta\boldsymbol{\phi}]$ is the *symplectic potential*, and $\boldsymbol{E}[\boldsymbol{\phi}]$ is a tensor-valued differential form[8] that encodes the equations of motion in the sense that, on shell, $\boldsymbol{E}[\boldsymbol{\phi}] = 0$. Thus, on shell, the integral of $\delta \boldsymbol{L}[\boldsymbol{\phi}]$ is just a boundary term, which we use to define $\boldsymbol{\theta}[\boldsymbol{\phi}; \delta\boldsymbol{\phi}]$. We can then define the *symplectic product* by taking a second, independent variation:

$$\boldsymbol{\omega}[\boldsymbol{\phi}; \delta_1\boldsymbol{\phi}, \delta_2\boldsymbol{\phi}] \equiv \delta_1\boldsymbol{\theta}[\boldsymbol{\phi}; \delta_2\boldsymbol{\phi}] - \delta_2\boldsymbol{\theta}[\boldsymbol{\phi}; \delta_1\boldsymbol{\phi}]. \tag{3.132}$$

From equation (3.131), it then follows that

$$\mathrm{d}\boldsymbol{\omega}[\boldsymbol{\phi}; \delta_1\boldsymbol{\phi}, \delta_2\boldsymbol{\phi}] = \delta_1\boldsymbol{E}[\boldsymbol{\phi}] \cdot \delta_2\boldsymbol{\phi} - \delta_2\boldsymbol{E}[\boldsymbol{\phi}] \cdot \delta_1\boldsymbol{\phi}, \tag{3.133}$$

which vanishes if $\delta_1\boldsymbol{\phi}$ and $\delta_2\boldsymbol{\phi}$ are both solutions to the linearized equations of motion. We define the corresponding vector current by

$${}_Sj^a\left[\boldsymbol{\phi}; \delta_1\boldsymbol{\phi}, \delta_2\boldsymbol{\phi}\right] \equiv \left({}^*\boldsymbol{\omega}\left[\boldsymbol{\phi}; \delta_1\boldsymbol{\phi}, \delta_2\boldsymbol{\phi}\right]\right)^a. \tag{3.134}$$

We now turn to two different Lagrangians whose symplectic products are particularly interesting. First, we consider the symplectic product for the Einstein-Hilbert Lagrangian four-form:

$$\boldsymbol{L}_{\mathrm{EH}}[\boldsymbol{g}] = \frac{1}{16\pi} R\boldsymbol{\epsilon}. \tag{3.135}$$

[8] Some of the indices of $\boldsymbol{E}[\boldsymbol{\phi}]$ are contracted with those of $\delta\boldsymbol{\phi}$, yielding a four-form $\boldsymbol{E}[\boldsymbol{\phi}] \cdot \delta\boldsymbol{\phi}$.

For this Lagrangian, we find (following [44], for example; note the difference in sign due to using a different sign convention for $R^a{}_{bcd}$)

$$(\theta_{\rm EH})_{abc}[\boldsymbol{g};\delta\boldsymbol{g}] = -\frac{1}{8\pi}\epsilon_{abcd}g^{fg}\delta^d{}_{[e}\delta C^e{}_{f]g}, \tag{3.136}$$

where $\delta C^a{}_{bc}$ is the variation of the connection coefficients for $\nabla_a(\lambda)$:

$$\delta C^a{}_{bc} = \frac{1}{2}g^{ad}(\nabla_b\delta g_{cd} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}). \tag{3.137}$$

Thus, the symplectic (vector) current is given by

$$\begin{aligned} \mathcal{S}j^a_{\rm EH}[\delta_1\boldsymbol{g},\delta_2\boldsymbol{g}] &= \frac{1}{8\pi}\delta^a{}_{[b}\delta_1 C^b{}_{c]d}\left[(\delta_2 g)^{cd} - \frac{1}{2}(\delta_2 g)^e{}_e g^{cd}\right] - 1 \longleftrightarrow 2 \\ &= \frac{1}{16\pi}\delta_1 C^a{}_{bc}(\delta_2 g)^{bc} + v^a[\delta_1\boldsymbol{g}](\delta_2 g)^b{}_b + w^{ab}[\delta_1\boldsymbol{g}]\nabla_b(\delta_2 g)^c{}_c - 1 \longleftrightarrow 2, \end{aligned} \tag{3.138}$$

for some tensor fields $v^a[\delta\boldsymbol{g}]$ and $w^{ab}[\delta\boldsymbol{g}]$ which are unimportant for the discussion of this chapter, as we only consider metric perturbations which are trace-free. Here, for simplicity, the dependence on the background metric g_{ab} is implicit. This symplectic product provides a bilinear current on the space of metric perturbations which is conserved for vacuum solutions to the linearized Einstein equations.

Somewhat unexpectedly, one can also define a symplectic product for the master variables. In order to do so, we need a Lagrangian formulation. As noted by Bini, Cherubini, Jantzen, and Ruffini [33], the Teukolsky operator can be rewritten as a modified wave operator:

$${}_s\Box = (\nabla^a + s\Gamma^a)(\nabla_a + s\Gamma_a) - 4s^2\Psi_2, \tag{3.139}$$

where

$$\Gamma^a = -2\left[\gamma l^a + (\epsilon+\rho)n^a - \alpha m^a - (\beta+\tau)\bar{m}^a\right]. \tag{3.140}$$

Since the equations of motion are now in the form of a modified wave equation, one can write down a Lagrangian four-form of the form (for $s \geq 0$)

$$\boldsymbol{L}_{\rm BCJR}[{}_s\Omega, {}_{-s}\Omega] = {}^*(\mathrm{d} + s\boldsymbol{\Gamma})\,{}_s\Omega \wedge (\mathrm{d} - s\boldsymbol{\Gamma})\,{}_{-s}\Omega - 96s^2\Psi_2\,{}_s\Omega\,{}_{-s}\Omega\boldsymbol{\epsilon}. \tag{3.141}$$

Note that, in this expression, the metric and Γ^a are non-dynamical fields, and therefore do not get varied. Varying this Lagrangian four-form results in the Teukolsky equations for spins s and $-s$.

One can easily show that

$$\boldsymbol{\theta}_{\mathrm{BCJR}}[\,{}_s\Omega,\,{}_{-s}\Omega;\delta\,{}_s\Omega,\delta\,{}_{-s}\Omega] = \delta\,{}_s\Omega^*(\mathrm{d}-s\boldsymbol{\Gamma})\,{}_{-s}\Omega + \delta\,{}_{-s}\Omega^*(\mathrm{d}+s\boldsymbol{\Gamma})\,{}_s\Omega, \tag{3.142}$$

and so

$$S j^a_{\mathrm{BCJR}}[\delta_1\,{}_s\Omega,\delta_1\,{}_{-s}\Omega;\delta_2\,{}_s\Omega,\delta_2\,{}_{-s}\Omega] = \delta_1\,{}_s\Omega(\nabla^a - s\Gamma^a)\delta_2\,{}_{-s}\Omega + \delta_1\,{}_{-s}\Omega(\nabla^a + s\Gamma^a)\delta_2\,{}_s\Omega - 1 \longleftrightarrow 2. \tag{3.143}$$

Here, we are dropping any dependence on the background values of ${}_s\Omega$ and ${}_{-s}\Omega$, since they do not appear on the right-hand side. Although this current is bilinear on the space of variations of the master variables, it can be regarded as a bilinear current on the space of master variables themselves, since their equations of motion are linear. Note further that this symplectic product is not the physical symplectic product for linearized gravity.

3.3.2 Currents of interest

Using the results of sections 3.2 and 3.3.1, we now define the following currents, for which we will be computing the geometric optics limit and the fluxes at the horizon and null infinity. The first of these currents is a rescaled version of the symplectic product of ${}_s\boldsymbol{\mathcal{C}}\cdot\delta\boldsymbol{g}$ and its complex conjugate:

$${}_{s\mathcal{C}}j^a[\delta\boldsymbol{g}] \equiv 8i\, S j^a_{\mathrm{EH}}\left[{}_s\boldsymbol{\mathcal{C}}\cdot\delta\boldsymbol{g},\,\overline{{}_s\boldsymbol{\mathcal{C}}\cdot\delta\boldsymbol{g}}\right], \tag{3.144}$$

in terms of the symplectic product (3.138) and the symmetry operator (3.74). The normalization here is chosen to give a nicer limit in geometric optics; similarly, this current is simpler in the limit of geometric optics than other currents that can be constructed from ${}_s\boldsymbol{\mathcal{C}}$. The currents defined in equation (3.144) are entirely local, but they generally diverge at null infinity, as we will show in section 3.5. The divergences can be removed by using ${}_s\mathring{\boldsymbol{\mathcal{C}}}$ instead of ${}_s\boldsymbol{\mathcal{C}}$. We therefore define

$${}_{2\mathring{\mathcal{C}}}j^a[\delta\boldsymbol{g}] \equiv 8i\sum_{s=\pm 2} S j^a_{\mathrm{EH}}\left[{}_s\mathring{\boldsymbol{\mathcal{C}}}\cdot\delta\boldsymbol{g},\,\overline{{}_s\mathring{\boldsymbol{\mathcal{C}}}\cdot\delta\boldsymbol{g}}\right], \tag{3.145}$$

where ${}_2\mathring{\boldsymbol{\mathcal{C}}}$ is defined in equation (3.128). The motivation for including the sum over s in this definition is due to the fact that ${}_2\mathring{\boldsymbol{\mathcal{C}}}$ and ${}_{-2}\mathring{\boldsymbol{\mathcal{C}}}$ are only nonzero for ingoing and outgoing solutions at null infinity, respectively. The sum therefore ensures that the total current is nonzero for both types of solutions.

We next define similar currents involving ${}_s\mathbf{X}$ and ${}_s\boldsymbol{\mathcal{D}}$:

$$ {}_{s\mathcal{D}}j^a[\delta \boldsymbol{g}] \equiv \frac{i}{16}\, s j^a_{\mathrm{EH}}\left[{}_s\mathbf{X}\cdot\delta\boldsymbol{g},\, \overline{{}_s\boldsymbol{\mathcal{D}}\cdot\delta\boldsymbol{g}}\right], \tag{3.146} $$

$$ {}_{2\mathring{\mathcal{D}}}j^a[\delta \boldsymbol{g}] \equiv \frac{i}{16}\sum_{s=\pm 2} s j^a_{\mathrm{EH}}\left[{}_s\mathring{\mathbf{X}}\cdot\delta\boldsymbol{g},\, \overline{{}_s\mathring{\boldsymbol{\mathcal{D}}}\cdot\delta\boldsymbol{g}}\right]. \tag{3.147} $$

Unlike the currents (3.144) and (3.145), both of these currents are nonlocal. We will see below that the geometric optics limits of these currents are proportional to the Carter constants K of the gravitons, as opposed to K^4 for the currents (3.144) and (3.145).

Finally, we define the currents

$$ {}_{s\Omega}j^a[\delta \boldsymbol{g}] \equiv \frac{1}{4\pi i}\, s j^a_{\mathrm{BCJR}}\left[{}_s\Omega,\, {}_{-s}\Omega;\, {}_{s,s}\widetilde{\mathcal{C}}\,\overline{{}_{-s}\Omega},\, {}_{-s,s}\widetilde{\mathcal{C}}\,\overline{{}_{-s}\Omega}\right], \tag{3.148} $$

in terms of the symplectic product for the master variables in equation (3.143) and the symmetry operator (3.73). Note that ${}_{\pm 2}\Omega$ are functions of δg_{ab}, by equation (3.31). These currents are very similar to the currents ${}_{\pm 2\mathcal{C}}j^a[\delta \boldsymbol{g}]$, having the same geometric optics limit, and also being local; however, these currents have the advantage of also having finite fluxes at null infinity.

The various properties of these currents are presented in table 3.1 and derived in sections 3.4 and 3.5. Note that, from table 3.1, only the currents ${}_{s\Omega}j^a$ are both local and possess finite fluxes.

3.4 Geometric Optics

Using the symmetry operators in section 3.2 and the symplectic products in section 3.3.1, one could define a multitude of currents that are conserved for vacuum solutions to the linearized Einstein equations. In this section, we provide the motivation for the particular currents highlighted in section 3.3.2. This is accomplished by taking the geometric optics limit, in which solutions to the linearized Einstein equations represent null fluids of gravitons. We express the associated currents in terms of the gravitons' constants of motion.

3.4.1 Basic formalism

The starting point for geometric optics is a harmonic ansatz for the metric perturbation:

$$ \delta g_{ab} = \mathrm{Re}\left\{\left[a\varpi_{ab} + O(\epsilon)\right]e^{-i\vartheta/\epsilon}\right\}, \tag{3.149} $$

where a and ϑ are real, ϖ_{ab}, the *polarization tensor*, is a complex, symmetric tensor that is normalized to satisfy $\varpi_{ab}\bar{\varpi}^{ab} = 1$, and ϵ is a dimensionless parameter whose limit is taken to zero. Inserting this ansatz into the linearized Einstein equations and the Lorenz gauge condition and equating coefficients of powers of ϵ yields the following results (see, for example, Misner, Thorne, and Wheeler [120]):

i. The wavevector k^a defined by

$$k_a \equiv \nabla_a \vartheta \tag{3.150}$$

is tangent to a congruence of null geodesics:

$$k^b \nabla_b k^a = 0, \qquad k_a k^a = 0. \tag{3.151}$$

ii. The polarization tensor ϖ_{ab} is orthogonal to k^a and parallel-transported along these geodesics:

$$k^a \varpi_{ab} = 0, \qquad k^c \nabla_c \varpi_{ab} = 0. \tag{3.152}$$

iii. The amplitude a evolves along these geodesics according to

$$\nabla_a (a^2 k^a) = 0. \tag{3.153}$$

We now consider this formalism in terms of spinors. First, as k^a is null, we can write

$$k^{AA'} = \kappa^A \bar{\kappa}^{A'}, \tag{3.154}$$

for some spinor κ^A. We choose a second spinor λ^A such that (κ, λ) form a spin basis. The conditions (3.152) and the normalization of ϖ_{ab} imply that

$$\varpi_{ab} = k_{(a}\alpha_{b)} + e_R q_a q_b + e_L \bar{q}_a \bar{q}_b, \tag{3.155}$$

where $q_a \equiv \kappa_A \bar{\lambda}_{A'}$ and α^a is an arbitrary vector satisfying $\alpha^a k_a = 0$. Because of the gauge freedom $\delta g_{ab} \to \delta g_{ab} + 2\nabla_{(a}\xi_{b)}$, the first term can be removed by a gauge transformation (which preserves the Lorenz gauge condition), and so we can safely set $\alpha^a = 0$.

The last two terms in equation (3.155) are physically measurable. The complex coefficients e_R and e_L correspond to right and left circular polarization. By the normalization of ϖ_{ab}, we have that $|e_R|^2 + |e_L|^2 = 1$. Moreover, these factors of e_R and e_L appear in the expansion for $(\delta\Psi)_{ABCD}$:

$$(\delta\Psi)_{ABCD} = -\frac{1}{\epsilon^2} a \kappa_A \kappa_B \kappa_C \kappa_D \left(e_R e^{-i\vartheta/\epsilon} + \bar{e}_L e^{i\vartheta/\epsilon} \right) + O(1/\epsilon). \tag{3.156}$$

3.4.2 Conserved currents

When considering nonlinear quantities in geometric optics, such as conserved currents, we will discard rapidly oscillating terms. This effectively takes a spacetime average of these quantities over a scale that is large compared to ϵ, but small compared to the radius of curvature of the background spacetime (see, for example, [101], or [43] for rigorous treatments of this averaging procedure via weak limits). Such an average we will denote by $\langle\cdot\rangle$.

Before considering the specific currents defined in section 3.3.2, we start with a few simple results. First, if a conserved current reduces in the limit of geometric optics to

$$\langle j^a\rangle = \frac{1}{\epsilon^n}[a^2 Q k^a + O(\epsilon)], \tag{3.157}$$

for some quantity Q and integer n, then Q is a conserved quantity along the integral curves of k^a. To see this, note that the leading order term in the conservation equation $\nabla_a\langle j^a\rangle = 0$ yields

$$0 = a^2 k^a \nabla_a Q + Q\nabla_a(a^2 k^a) = a^2 k^a \nabla_a Q, \tag{3.158}$$

from equation (3.153). All currents that we consider in this chapter will be of the form (3.157) in the geometric optics limit.

The second result is that, under the assumption (3.157), the conserved charge associated with the current j^a reduces to a sum over all gravitons of the conserved quantity Q for each graviton. This result means that equation (3.157) is a physically appealing assumption. The proof proceeds as follows [120]: first, we note that the effective stress-energy tensor appropriate to gravitational radiation in the geometric optics regime is given by [101]

$$\langle T^{\text{eff}}_{ab}\rangle = \frac{1}{32\pi}\left\langle(\nabla_a\delta g_{cd})[\nabla_b(\delta g)^{cd}]\right\rangle + O(1/\epsilon) = \frac{a^2}{32\pi\epsilon^2}\left[k_a k_b + O(\epsilon)\right]. \tag{3.159}$$

On the other hand, the stress-energy tensor for a collection of gravitons with number-flux $\mathcal{N}_a$ and momentum $p_a = \hbar k_a/\epsilon$ is given by [120]

$$T^{\text{eff}}_{ab} = p_{(a}\mathcal{N}_{b)}, \tag{3.160}$$

and so we find that

$$a^2 k_a = 32\pi\hbar\epsilon\mathcal{N}_a[1 + O(\epsilon)]. \tag{3.161}$$

Upon integrating a current j^a given by equation (3.157) over a hypersurface Σ, one finds the charge

$$\int_\Sigma \langle j^a \rangle \mathrm{d}^3\Sigma_a = \frac{32\pi\hbar}{\epsilon^{n-1}} \sum_\alpha Q_\alpha [1 + O(\epsilon)], \tag{3.162}$$

where α labels the gravitons passing through the hypersurface. That is, the charge is proportional to the sum of the conserved quantities over all of the gravitons passing through the surface.

3.4.3 Computations

We now turn to computations of geometric optics limits for the conserved currents discussed in this chapter. For these calculations, we first define the quantities κ_0, κ_1, r_a, and s_a:

$$\kappa_0 \equiv o_A \kappa^A, \qquad \kappa_1 \equiv \iota_A \kappa^A, \qquad r^a \equiv \sigma^a{}_{AA'} o^A \bar{\kappa}^{A'}, \qquad s^a \equiv \sigma^a{}_{AA'} \iota^A \bar{\kappa}^{A'}. \tag{3.163}$$

These quantities are constructed from the spinor κ_A (which is related to the wavevector k^a) and the principal spin basis (o, ι). They satisfy

$$\begin{gathered} |\zeta \kappa_0 \kappa_1|^2 = \frac{\epsilon^2}{2\hbar^2} K, \qquad r_a r^a = s_a s^a = r_a k^a = s_a k^a = 0, \\ r_a \bar{r}^a = |\kappa_0|^2, \qquad s_a \bar{s}^a = |\kappa_1|^2, \qquad r_a \bar{s}^a = -\kappa_0 \bar{\kappa}_1, \end{gathered} \tag{3.164}$$

where $K = \hbar^2 K_{ab} k^a k^b / \epsilon^2$ is the Carter constant for the gravitons. The factors of $\hbar$ arise in this classical computation in the conversion from the wavevectors of the gravitons to their momenta, and hence their conserved quantities.

We now begin calculating the conserved currents defined in section 3.3.2. Since, to leading order in geometric optics, the differential operators present in this chapter become c-numbers, a straightforward calculation starting from equations (3.32) and (3.69) shows that

$${}_s\mathcal{T}^\dagger_{ab} = \frac{1}{\epsilon^2} \begin{cases} \kappa_0^2 r_a r_b + O(\epsilon) & s = 2 \\ \zeta^4 \kappa_1^2 s_a s_b + O(\epsilon) & s = -2 \end{cases}, \tag{3.165a}$$

$${}_sM^{ab} = \frac{1}{2\epsilon^2} \begin{cases} \kappa_0^2 r^a r^b + O(\epsilon) & s = 2 \\ \zeta^4 \kappa_1^2 s^a s^b + O(\epsilon) & s = -2 \end{cases}, \tag{3.165b}$$

and [starting from equation (3.156)] that

$${}_s\Omega = -\frac{a}{\epsilon^2} (e_R e^{-i\vartheta/\epsilon} + \bar{e}_L e^{i\vartheta/\epsilon}) \begin{cases} \kappa_0^4 + O(\epsilon) & s = 2 \\ (\zeta\kappa_1)^4 + O(\epsilon) & s = -2 \end{cases}. \tag{3.166}$$

As such, we find that

$$ {}_s\mathcal{C}_{ab}{}^{cd}\delta g_{cd} = -\frac{a}{\epsilon^4}\zeta^4(\kappa_1\kappa_0)^2(e_R e^{-i\vartheta/\epsilon} + \bar{e}_L e^{i\vartheta/\epsilon}) \begin{cases} r_a r_b \kappa_1^2 + O(\epsilon) & s = 2 \\ s_a s_b \kappa_0^2 + O(\epsilon) & s = -2 \end{cases}. \tag{3.167} $$

This implies that

$$ \left\langle ({}_s\mathcal{C}_{bc}{}^{de}\delta g_{de})\nabla^a \overline{{}_s\mathcal{C}^{bc}{}_{de}\delta g^{de}} \right\rangle = -\frac{2\pi i}{\hbar^7}K^4(|e_R|^2 - |e_L|^2)\mathcal{N}^a[1 + O(\epsilon)]. \tag{3.168} $$

Thus, we find that the current ${}_{s\mathcal{C}}j^a[\delta\boldsymbol{g}]$ is given in this limit by

$$ \begin{aligned} \langle {}_{s\mathcal{C}}j^a[\delta\boldsymbol{g}]\rangle &= \frac{1}{2\pi}\left\langle \mathrm{Im}\left[({}_s\mathcal{C}_{bc}{}^{de}\delta g_{de})\nabla^a \overline{{}_s\mathcal{C}^{bc}{}_{de}\delta g^{de}}\right]\right\rangle [1 + O(\epsilon)] \\ &= \frac{1}{\hbar^7}K^4\left(|e_R|^2 - |e_L|^2\right)\mathcal{N}^a[1 + O(\epsilon)]. \end{aligned} \tag{3.169} $$

As such, these currents are a generalization of the Carter constant for point particles to linearized gravity in the Kerr spacetime, at least in the limit of geometric optics.

We now turn to the current ${}_{s\mathcal{D}}j^a[\delta\boldsymbol{g}]$. First, note that, from equations (3.63) and (3.49),

$$ {}_s\mathcal{D}\,{}_s\Omega = \frac{1}{\epsilon^2}|\zeta\kappa_0\kappa_1|^2\,{}_s\Omega[1 + O(\epsilon)], \tag{3.170} $$

and so

$$ {}_s\mathcal{D}_{ab}{}^{cd}\delta g_{cd} = \frac{K}{2\hbar^2}\,{}_s\mathrm{X}_{ab}{}^{cd}\delta g_{cd}[1 + O(\epsilon)]. \tag{3.171} $$

Now, note that ${}_s\mathrm{X}_{ab}{}^{cd}\delta g_{cd}$, by equations (3.108) and (3.105), can be written (in the limit of geometric optics, where differential operators commute to leading order) as a product of the form

$$ {}_s\mathrm{X}_{ab}{}^{cd}\delta g_{cd} = 4\left({}_{s,s}\tilde{\mathcal{C}}\,\overline{{}_{-s,-s}\tilde{\mathcal{C}}}\right)^{-1}\,{}_s\mathcal{C}_{ab}{}^{cd}\,\overline{{}_{-s}\mathcal{C}_{cd}{}^{ef}}\delta g_{ef}[1 + O(\epsilon)], \tag{3.172} $$

where the operator $\left({}_{s,s}\tilde{\mathcal{C}}\,\overline{{}_{-s,-s}\tilde{\mathcal{C}}}\right)^{-1}$ is a nonlocal operator having the effect of multiplying each coefficient of the expansion (3.58) by $64/(C^2_{lm\omega} + 144M^2\omega^2)$. This operator is a nonlocal inverse to ${}_{s,s}\tilde{\mathcal{C}}\,\overline{{}_{-s,-s}\tilde{\mathcal{C}}}$, by equation (3.94). For its geometric optics limit, note that

$$ {}_{2,-2}\widetilde{\mathcal{C}}\,\overline{{}_2\Omega} = \frac{1}{2\epsilon^4}(\bar{\zeta}\kappa_0\bar{\kappa}_1)^4\overline{{}_2\Omega}[1 + O(\epsilon)], \qquad {}_{-2,2}\widetilde{\mathcal{C}}\,\overline{{}_{-2}\Omega} = \frac{1}{2\epsilon^4}(\zeta\bar{\kappa}_0\kappa_1)^4\overline{{}_{-2}\Omega}[1 + O(\epsilon)], \tag{3.173a} $$

$$ {}_{2,2}\widetilde{\mathcal{C}}\,\overline{{}_{-2}\Omega} = \frac{1}{2\epsilon^4}|\kappa_0|^8\overline{{}_{-2}\Omega}[1 + O(\epsilon)], \qquad {}_{-2,-2}\widetilde{\mathcal{C}}\,\overline{{}_2\Omega} = \frac{1}{2\epsilon^4}|\zeta\kappa_1|^8\overline{{}_2\Omega}[1 + O(\epsilon)], \tag{3.173b} $$

and so

$$ \left({}_{s,s}\tilde{\mathcal{C}}\,\overline{{}_{-s,-s}\tilde{\mathcal{C}}}\right)^{-1}\,{}_s\Omega = \frac{4\epsilon^8}{|\zeta\kappa_0\kappa_1|^8}\,{}_s\Omega[1 + O(\epsilon)]. \tag{3.174} $$

Moreover, we have that [from equations (3.165a) and (3.165b)]

$$ {}_s\mathcal{C}_{ab}{}^{cd}\overline{{}_{-s}\mathcal{C}_{cd}{}^{ef}}\delta g_{ef} = -\frac{a}{4\epsilon^8}|\zeta\kappa_0\kappa_1|^8(\bar{e}_R e^{i\vartheta/\epsilon} + e_L e^{-i\vartheta/\epsilon})\begin{cases} r_a r_b/\kappa_0^2 + O(\epsilon) & s=2 \\ s_a s_b/\kappa_1^2 + O(\epsilon) & s=-2 \end{cases}, \tag{3.175} $$

from which it follows that

$$ {}_s\mathrm{X}_{ab}{}^{cd}\delta g_{cd} = -4a(\bar{e}_R e^{i\vartheta/\epsilon} + e_L e^{-i\vartheta/\epsilon})\begin{cases} r_a r_b/\kappa_0^2 + O(\epsilon) & s=2 \\ s_a s_b/\kappa_1^2 + O(\epsilon) & s=-2 \end{cases}. \tag{3.176} $$

The current in question is then given by

$$ \langle {}_{s\mathcal{D}}j^a[\delta\boldsymbol{g}]\rangle = \frac{1}{\hbar}K\left(|e_R|^2 - |e_L|^2\right)\mathcal{N}^a[1+O(\epsilon)]. \tag{3.177} $$

This therefore provides another, entirely *non-local* notion of the Carter constant for linearized gravity in the Kerr spacetime.

There are, of course, other currents whose charges reduce to the Carter constant in the geometric optics limit. Another class of currents come from the symplectic product for the master variables, instead of the metric perturbation. One current of interest from this class is given by equation (3.148), which has a limit in geometric optics given by [from equations (3.143), (3.173), and (3.166)]

$$ \langle {}_{s\Omega}j^a[\delta\boldsymbol{g}]\rangle = \frac{1}{\hbar^7}K^4(|e_R|^2 - |e_L|^2)\mathcal{N}^a[1+O(\epsilon)]. \tag{3.178} $$

The results of this section [equations (3.169), (3.177), and (3.178)] give the expressions in table 3.1, at least for the currents that do not involve projection operators. We now consider the two remaining currents, ${}_{2\mathring{\mathcal{C}}}j^a[\delta\boldsymbol{g}]$ and ${}_{2\mathring{\mathcal{D}}}j^a[\delta\boldsymbol{g}]$. For simplicity, we first consider ${}_{2\mathring{\mathcal{C}}}j^a[\delta\boldsymbol{g}]$ (the exact same argument holds for ${}_{2\mathring{\mathcal{D}}}j^a[\delta\boldsymbol{g}]$). This current is the sum of two terms, the first of which is equal to ${}_{-2\mathcal{C}}j^a[\delta\boldsymbol{g}]$, except that it contains a projection which eliminates the ingoing modes at null infinity. Similarly, the second term is equal to ${}_{2\mathcal{C}}j^a[\delta\boldsymbol{g}]$, except it eliminates all outgoing modes. Consider the case where δg_{ab} represents a null fluid of gravitons where the gravitons are purely outgoing at future null infinity; that is, k^a is tangent to an outgoing null congruence. The geometric optics limit in this case would be the same as that of ${}_{-2\mathcal{C}}j^a[\delta\boldsymbol{g}]$. Similarly, if k^a is an ingoing null congruence, the geometric optics limit would be the same as that of ${}_{2\mathcal{C}}j^a[\delta\boldsymbol{g}]$. Since these geometric optics limits are equal by equation (3.169), we recover the result in table 3.1:

$$\left\langle {}_{2}\mathring{\mathcal{C}}j^a[\delta\boldsymbol{g}] \right\rangle = \frac{1}{\hbar^7} K^4 \left(|e_R|^2 - |e_L|^2\right) \mathcal{N}^a [1 + O(\epsilon)], \tag{3.179}$$

when δg_{ab} represents either an ingoing or outgoing null fluid of gravitons. A similar argument gives an analogous result for ${}_{2}\mathring{\mathcal{D}}j^a[\delta\boldsymbol{g}]$. However, the geometric optics limits for ${}_{2}\mathring{\mathcal{C}}j^a[\delta\boldsymbol{g}]$ and ${}_{2}\mathring{\mathcal{D}}j^a[\delta\boldsymbol{g}]$ are only given by simple expressions when k^a is either tangent to an ingoing or outgoing null congruence, but not for general geometric optics solutions δg_{ab}.

We conclude this discussion with a brief review of a classification scheme for conserved currents in geometric optics that we used in [82]. In the limit of geometric optics, one often finds that conserved currents depend on the quantities e_R and e_L in one of the following four ways, and (depending on which dependence they have) these currents are called either *energy*, *zilch*, *chiral*, or *antichiral currents*:

$$\langle j^a \rangle = Q\mathcal{N}^a \begin{cases} 1 + O(\epsilon) & \text{energy currents} \\ (|e_R|^2 - |e_L|^2) + O(\epsilon) & \text{zilch currents} \\ e_R \bar{e}_L + O(\epsilon) & \text{chiral currents} \\ \bar{e}_R e_L + O(\epsilon) & \text{antichiral currents} \end{cases}. \tag{3.180}$$

This classification scheme is a specialization of that of [13]. For conserved currents that are $\mathbb{R}$-bilinear functionals of $(\delta\Psi)_{ABCD}$ (a property which is satisfied by all currents considered in this chapter), there is a relationship between Q and the type of current in this classification: for energy and zilch currents,

$$Q = Q_{a_1 \cdots a_n} p^{a_1} \cdots p^{a_n}, \tag{3.181}$$

where $Q_{a_1 \cdots a_n}$ is a rank n Killing tensor, where n is odd for energy currents and even for zilch currents. Moreover, for chiral and antichiral currents, Q cannot be written in the above form. Since we wanted to construct conserved currents which were related to the Carter constant, which is a conserved quantity arising from a rank two Killing tensor, it is unsurprising that all currents which we considered were zilch currents.

Another interesting result of this classification scheme is the following odd property of the symplectic product for the master variables. The symplectic product for linearized gravity, when applied to δg_{ab} and $\pounds_\xi \delta g_{ab}$, gives an energy current in geometric optics, and the associated conserved quantity is proportional to $\xi^a p_a$ (which would be proportional to the energy in the case $\xi^a = t^a$). This current is known as the *canonical energy current*. However, using the symplectic product for the master variables, one finds that a similar current, obtained by using ${}_{\pm s}\Omega$ and $\pounds_\xi\, {}_{\pm s}\Omega$, gives a chiral current. In this sense, the symplectic product for the master variables cannot be used to construct a current whose geometric optics limit behaves like energy.

3.5 Fluxes through Null Infinity and the Horizon

Another desirable property for a conserved current is that its flux through the horizon (H) and through null infinity ($\mathscr{I}$) be finite. In this section, we first define these fluxes in section 3.5.1, and then consider the asymptotic falloffs of the fields that occur in these fluxes in section 3.5.2. The final values of our fluxes are given in section 3.5.3.

3.5.1 Integration along the horizon and null infinity

We begin with the definition of these fluxes. First, the Boyer-Lindquist coordinate system is not well suited to working at the horizon or null infinity. Instead, one uses the ingoing and outgoing coordinate systems (v, r, θ, ψ) and (u, r, θ, χ), defined in terms of Boyer-Lindquist coordinates and the tortoise coordinate (3.115) by

$$v = t + r^*, \qquad \psi = \phi + \int \frac{a dr}{\Delta}, \tag{3.182a}$$

$$u = t - r^*, \qquad \chi = \phi - \int \frac{a dr}{\Delta}. \tag{3.182b}$$

The ingoing coordinate system is relevant near the future horizon (H^+) and past null infinity ($\mathscr{I}^-$), while the outgoing coordinate system is relevant near the past horizon (H^-) and future null infinity ($\mathscr{I}^+$). When dealing with a generic surface S, we will write w and α instead of either v and ψ or u and χ:

$$w = \begin{cases} v & \text{at } H^+,\ \mathscr{I}^- \\ u & \text{at } H^-,\ \mathscr{I}^+ \end{cases}, \qquad \alpha = \begin{cases} \psi & \text{at } H^+,\ \mathscr{I}^- \\ \chi & \text{at } H^-,\ \mathscr{I}^+ \end{cases}. \tag{3.183}$$

This greatly simplifies definitions.

The flux of a current ${}_{\dots}j^a$ through a surface S of constant r (such as the horizon or null infinity) is defined by

$$\left.\frac{\mathrm{d}^2\,{}_{\dots}Q}{\mathrm{d}w\mathrm{d}\Omega}\right|_S \equiv \lim_{\to S}(r^2+a^2)\,{}_{\dots}j^a N_a, \tag{3.184}$$

where $\mathrm{d}\Omega \equiv \sin\theta\mathrm{d}\theta\mathrm{d}\alpha$ is the differential solid angle, N_a is the surface normal, and the factor of r^2+a^2 comes from the fact that the determinant of the induced metric on surfaces of constant r is $(r^2+a^2)\sin\theta$. The surface normals are proportional to $(\mathrm{d}r)_a$,

$$N_a \propto (\mathrm{d}r)_a = n_a - \frac{\Delta}{2\Sigma}l_a, \tag{3.185}$$

and the usual scaling freedom is fixed by requiring[9] that either $N^a\nabla_a u = 1$ (for H^- and $\mathscr{I}^+$) or $N^a\nabla_a v = 1$ (for H^+ and $\mathscr{I}^-$). It turns out, however, that these requirements are the same, and fix the normalization such that

$$N_a = \frac{1}{r^2+a^2}\left(\Sigma n_a - \frac{\Delta}{2}l_a\right). \tag{3.186}$$

As such, we find that

$$\left.\frac{\mathrm{d}^2Q}{\mathrm{d}v\mathrm{d}\Omega}\right|_{H^+} = \lim_{r\to r_+, v\text{ fixed}} \Sigma\left(j_n - \frac{\Delta}{2\Sigma}j_l\right), \tag{3.187a}$$

$$\left.\frac{\mathrm{d}^2Q}{\mathrm{d}u\mathrm{d}\Omega}\right|_{H^-} = \lim_{r\to r_+, u\text{ fixed}} \Sigma\left(j_n - \frac{\Delta}{2\Sigma}j_l\right), \tag{3.187b}$$

$$\left.\frac{\mathrm{d}^2Q}{\mathrm{d}v\mathrm{d}\Omega}\right|_{\mathscr{I}^-} = \lim_{r\to \infty, v\text{ fixed}} r^2\left(j_n - \frac{1}{2}j_l\right), \tag{3.187c}$$

$$\left.\frac{\mathrm{d}^2Q}{\mathrm{d}u\mathrm{d}\Omega}\right|_{\mathscr{I}^+} = \lim_{r\to \infty, v\text{ fixed}} r^2\left(j_n - \frac{1}{2}j_l\right), \tag{3.187d}$$

where l, n, m, and $\bar{m}$ subscripts denote contraction on an index with the corresponding member of the null tetrad.

From this discussion, for the calculations in section 3.5.3, we will need the components of symplectic products along l_a and n_a. To determine the values of these components, we first note that we compute fluxes of the conserved currents (3.144), (3.145), (3.146), and (3.147) *only* when acting upon the metric perturbations $\mathrm{Im}[\check{\delta}_\pm g_{ab}]$. We are free to do so, as these metric perturbations are

[9] Note that, if one were integrating these currents on a finite portion of these surfaces, the normalization of N_a would not matter. However, for equation (3.184) to hold—that is, when integrating over an infinitesimal portion $\mathrm{d}w$, for $w = u$ or v—we must normalize N_a appropriately.

related by a gauge transformation to any $l \geq 2$ metric perturbation δg_{ab}. Moreover, this specialization allows us to use equations (3.109) and (3.114) in order to write the fluxes in terms of the fluxes of the currents

$$ {}_{\pm 2}j^a_{ll'm\omega pp'} \equiv {}_S j^a_{\mathrm{EH}}\left[(\delta_{\pm}\boldsymbol{g})_{lm\omega p}, \overline{(\delta_{\pm}\boldsymbol{g})_{l'm\omega p'}}\right], \tag{3.188} $$

assuming that we average our fluxes over w and α. As such, we note that

$$ {}_2(j_{ll'm\omega pp'})_l = -\frac{1}{16\pi}\left[(\delta_+ C_{lm\omega p})_{l\bar{m}\bar{m}}\overline{(\delta_+ g_{l'm\omega p'})^{\bar{m}\bar{m}}} - \overline{l, p \longleftrightarrow l', p'}\right], \tag{3.189a} $$

$$ \begin{aligned} {}_2(j_{ll'm\omega pp'})_n = -\frac{1}{16\pi}\Big[&(\delta_+ C_{lm\omega p})_{n\bar{m}\bar{m}}\overline{(\delta_+ g_{l'm\omega p'})^{\bar{m}\bar{m}}} - (\delta_+ C_{lm\omega p})_{n(l\bar{m})}\overline{(\delta_+ g_{l'm\omega p'})^{(n\bar{m})}} \\ &- \overline{l, p \longleftrightarrow l', p'}\Big], \end{aligned} \tag{3.189b} $$

where $\overline{l, p \longleftrightarrow l', p'}$ means "the preceding terms, complex conjugated and with l and p switched with l' and p'". The non-zero perturbed connection coefficients are given by

$$ (\delta_+ C)_{l\bar{m}\bar{m}} = -\frac{1}{2}[D + 2(\epsilon - \bar{\epsilon}) - \rho](\delta_+ g)_{\bar{m}\bar{m}}, \tag{3.190a} $$

$$ (\delta_+ C)_{n(l\bar{m})} = -\frac{1}{4}(D + 2\epsilon + \rho)(\delta_+ g)_{(n\bar{m})} - \frac{1}{2}\tau(\delta_+ g)_{\bar{m}\bar{m}}, \tag{3.190b} $$

$$ (\delta_+ C)_{n\bar{m}\bar{m}} = -\frac{1}{4}(\delta + 2\bar{\alpha})(\delta_+ g)_{(n\bar{m})} - \frac{1}{2}[\Delta + 2(\gamma - \bar{\gamma}) - 2\mu](\delta_+ g)_{\bar{m}\bar{m}}. \tag{3.190c} $$

One can obtain the analogous expressions for δ_-, which defines ${}_{-2}j_{ll'm\omega pp'}$, by performing a $'$ transformation. For the symplectic product defined using the master variables, we find that

$$ {}_S j_l^{\mathrm{BCJR}}[\delta_1\, {}_s\Omega, \delta_1\, {}_{-s}\Omega; \delta_2\, {}_s\Omega, \delta_2\, {}_{-s}\Omega] = \delta_1\, {}_s\Omega(D - s\Gamma_l)\delta_2\, {}_{-s}\Omega + \delta_1\, {}_{-s}\Omega(D + s\Gamma_l)\delta_2\, {}_s\Omega - 1 \longleftrightarrow 2, \tag{3.191a} $$

$$ {}_S j_n^{\mathrm{BCJR}}[\delta_1\, {}_s\Omega, \delta_1\, {}_{-s}\Omega; \delta_2\, {}_s\Omega, \delta_2\, {}_{-s}\Omega] = \delta_1\, {}_s\Omega(\Delta - s\Gamma_n)\delta_2\, {}_{-s}\Omega + \delta_1\, {}_{-s}\Omega(\Delta + s\Gamma_n)\delta_2\, {}_s\Omega - 1 \longleftrightarrow 2. \tag{3.191b} $$

3.5.2 Asymptotic behavior of the relevant fields

In order to determine fluxes at null infinity and the horizon, we also need to know the asymptotic behavior of the quantities that appear in equation (3.189) and its $'$ transform. This asymptotic behavior can be captured in the following way: first define, for some quantity $q[{}_s\psi]$, with coefficients $q_{lm\omega p}[{}_s\psi]$ in an expansion, the falloff rates $n_q^{\mathrm{in/out/down/up}}$ and the angular dependences

${}_qS^{\text{in/out/down/up}}_{lm\omega p}(\theta)$ by

$$q_{lm\omega p}(t,r,\theta,\phi) \equiv \begin{cases} {}_s\widehat{\psi}^{\text{in}}_{lm\omega p} e^{i(m\psi-\omega v)}\, {}_qS^{\text{in}}_{lm\omega p}(\theta)\Delta^{n_q^{\text{in}}} + {}_s\widehat{\psi}^{\text{out}}_{lm\omega p} e^{i(m\chi-\omega u)}\, {}_qS^{\text{out}}_{lm\omega p}(\theta)\Delta^{n_q^{\text{out}}} & r\to r_+ \\ {}_s\widehat{\psi}^{\text{down}}_{lm\omega p} e^{i(m\psi-\omega v)}\, {}_qS^{\text{down}}_{lm\omega p}(\theta) r^{n_q^{\text{down}}} + {}_s\widehat{\psi}^{\text{up}}_{lm\omega p} e^{i(m\chi-\omega u)}\, {}_qS^{\text{up}}_{lm\omega p}(\theta) r^{n_q^{\text{up}}} & r\to\infty \end{cases}. \tag{3.192}$$

Assuming appropriate smoothness conditions, equation (3.192) simplifies further if we specialize to the various surfaces at which we are computing these quantities:

$$q_{lm\omega p}(t,r,\theta,\phi)\big|_S \sim \begin{cases} {}_s\widehat{\psi}^{\text{in}}_{lm\omega p} e^{i(m\psi-\omega v)}\, {}_qS^{\text{in}}_{lm\omega p}(\theta)\Delta^{n_q^{\text{in}}} & S = H^+ \\ {}_s\widehat{\psi}^{\text{out}}_{lm\omega p} e^{i(m\chi-\omega u)}\, {}_qS^{\text{out}}_{lm\omega p}(\theta)\Delta^{n_q^{\text{out}}} & S = H^- \\ {}_s\widehat{\psi}^{\text{down}}_{lm\omega p} e^{i(m\psi-\omega v)}\, {}_qS^{\text{down}}_{lm\omega p}(\theta) r^{n_q^{\text{down}}} & S = \mathscr{I}^- \\ {}_s\widehat{\psi}^{\text{up}}_{lm\omega p} e^{i(m\chi-\omega u)}\, {}_qS^{\text{up}}_{lm\omega p}(\theta) r^{n_q^{\text{up}}} & S = \mathscr{I}^+ \end{cases}. \tag{3.193}$$

In other words, only "in" modes contribute at H^+, "out" modes at H^-, etc. The various quantities q which we will be considering will be components of metric perturbations and perturbed connection coefficients that occur in equation (3.189) and its $'$ transform.

To determine these falloff rates and asymptotic angular dependences, we first write the quantities that appear in (3.189) and its $'$ transform in terms of differential operators acting upon the Debye potential, using the operators defined in equation (3.42): the perturbed metric satisfies

$$(\eth_+ g)_{(n\bar{m})} = -\frac{1}{\sqrt{2}\bar{\zeta}}\Bigg[\left(\mathscr{D}_0 + \frac{1}{\zeta} - \frac{2}{\bar{\zeta}}\right)\left(\mathscr{L}_2^+ - \frac{3ia\sin\theta}{\zeta}\right) + \left(\mathscr{L}_2^+ + \frac{ia\sin\theta}{\zeta} + \frac{2ia\sin\theta}{\bar{\zeta}}\right)\left(\mathscr{D}_0 - \frac{3}{\zeta}\right)\Bigg]\, {}_{-2}\psi, \tag{3.194a}$$

$$(\eth_+ g)_{\bar{m}\bar{m}} = -\left(\mathscr{D}_0 + \frac{1}{\zeta}\right)\left(\mathscr{D}_0 - \frac{3}{\zeta}\right)\, {}_{-2}\psi, \tag{3.194b}$$

$$(\eth_- g)_{(lm)} = \frac{\zeta^2}{2\sqrt{2}\bar{\zeta}\Delta}\Bigg[\left(\mathscr{L}_2 + \frac{ia\sin\theta}{\zeta} + \frac{2ia\sin\theta}{\bar{\zeta}}\right)\left(\mathscr{D}_0^+ - \frac{3}{\zeta}\right) + \left(\mathscr{D}_0^+ + \frac{1}{\zeta} - \frac{2}{\bar{\zeta}}\right)\left(\mathscr{L}_2 - \frac{3ia\sin\theta}{\zeta}\right)\Bigg]\Delta^2\, {}_2\psi, \tag{3.194c}$$

$$(\eth_- g)_{mm} = \frac{\zeta^2}{4\bar{\zeta}^2}\left(\mathscr{D}_0^+ + \frac{1}{\zeta}\right)\left(\mathscr{D}_0^+ - \frac{3}{\zeta}\right)\Delta^2\, {}_2\psi, \tag{3.194d}$$

whereas the relevant perturbed connection coefficients are given by

$$(\eth_+ C)_{l\bar{m}\bar{m}} = -\frac{1}{2}\left(\mathscr{D}_0 + \frac{1}{\zeta}\right)(\eth_+ g)_{\bar{m}\bar{m}}, \tag{3.195a}$$

$$(\eth_+ C)_{n(l\bar{m})} = -\frac{1}{4}\left(\mathscr{D}_0 - \frac{1}{\zeta}\right)(\eth_+ g)_{(n\bar{m})} + \frac{ia\sin\theta}{2\sqrt{2}\Sigma}(\eth_+ g)_{\bar{m}\bar{m}}, \tag{3.195b}$$

$$(\eth_+ C)_{n\bar{m}\bar{m}} = -\frac{1}{4\sqrt{2}\bar{\zeta}}\left(\mathscr{L}_{-1}^{+} - \frac{2ia\sin\theta}{\bar{\zeta}}\right)(\eth_+ g)_{(n\bar{m})} + \frac{\Delta}{4\Sigma}\left(\mathscr{D}_0^{+} - \frac{2}{\zeta} - \frac{2}{\bar{\zeta}}\right)(\eth_+ g)_{\bar{m}\bar{m}}, \tag{3.195c}$$

$$(\eth_- C)_{nmm} = \frac{\Delta}{4\Sigma}\left(\mathscr{D}_0^{+} - \frac{1}{\zeta} + \frac{2}{\bar{\zeta}}\right)(\eth_- g)_{mm}, \tag{3.195d}$$

$$(\eth_- C)_{l(nm)} = \frac{1}{8\Sigma}\left(\mathscr{D}_0^{+} - \frac{3}{\zeta}\right)\Delta(\eth_- g)_{(lm)} + \frac{ia\sin\theta}{2\sqrt{2}\zeta^2}(\eth_- g)_{mm}, \tag{3.195e}$$

$$(\eth_- C)_{lmm} = -\frac{1}{4\sqrt{2}\zeta}\mathscr{L}_{-1}(\eth_- g)_{(lm)} - \frac{1}{2}\left(\mathscr{D}_0 - \frac{2}{\zeta}\right)(\eth_- g)_{mm}. \tag{3.195f}$$

In order to compute the asymptotic behavior of these quantities, one needs to determine the asymptotic behavior of derivatives of the Debye potential. However, applying the naïve approach, which uses the asymptotic expansions given by equations (3.116) and (3.118), along with equation (3.123), results in cancellations in the leading-order behavior. Instead, we use the radial Teukolsky-Starobinsky identity (3.92), which provides a differential equation that is independent of the radial Teukolsky equation (3.51b). Using the radial Teukolsky equation, one can reduce the radial Teukolsky-Starobinsky identity to the following expression for derivatives of ${}_s\widehat{\Omega}_{lm\omega p}(r)$ [52]:

$$\mathscr{D}_{0(\mp m)(\mp\omega)}\Delta^{(2\pm2)/2}\,{}_{\pm2}\widehat{\Omega}_{lm\omega p} \equiv {}_{\pm2}\Xi_{lm\omega p}\Delta^{(2\pm2)/2}\,{}_{\pm2}\widehat{\Omega}_{lm\omega p} + {}_{\pm2}\Pi_{lm\omega p}\Delta^{(2\mp2)/2}\,{}_{\mp2}\widehat{\Omega}_{lm\omega p}, \tag{3.196}$$

where this equation defines the coefficients ${}_{\pm s}\Xi_{lm\omega p}$ and ${}_{\pm s}\Pi_{lm\omega p}$. These equations also clearly hold for ${}_s\widehat{\psi}_{lm\omega p}(r)$.

Plugging equation (3.196) [for ${}_s\psi_{lm\omega p}(r)$] into equations (3.194) and (3.195), and then taking the limits $r \to \infty$ and $r \to r_+$, yields the asymptotic falloffs given in table 3.2. Using this same calculation, we can determine the angular dependences of the quantities in (3.194) and (3.195). Defining, for $s \geq 0$,

$${}_{\pm s}\eta^{+}_{lm\omega} = \pm2i(2s-1)\omega r_+ - {}_2\lambda_{lm\omega}, \qquad {}_{\pm s}\eta^{\infty}_{lm\omega} = \pm2(2s-1)\omega a\cos\theta + {}_2\lambda_{lm\omega}, \tag{3.197}$$

Table 3.2: Asymptotic behavior of the solutions for linearized gravity.

	Ingoing $[e^{i(m\psi-\omega v)}\times]$		Outgoing $[e^{i(m\chi-\omega u)}\times]$	
	$r\to r_+$	$r\to\infty$	$r\to r_+$	$r\to\infty$
$(\eth_+ g_{lm\omega p})_{n\bar{m}}$	Δ	$1/r^2$	1	r
$(\eth_+ g_{lm\omega p})_{\bar{m}\bar{m}}$	1	$1/r$	1	1
$(\eth_- g_{lm\omega p})_{lm}$	$1/\Delta$	r	1	$1/r^2$
$(\eth_- g_{lm\omega p})_{mm}$	1	1	1	$1/r$
$(\eth_+ C_{lm\omega p})_{l\bar{m}\bar{m}}$	$1/\Delta$	$1/r$	1	$1/r$
$(\eth_+ C_{lm\omega p})_{n(l\bar{m})}$	1	$1/r^2$	1	$1/r^2$
$(\eth_+ C_{lm\omega p})_{n\bar{m}\bar{m}}$	Δ	$1/r^2$	1	1
$(\eth_- C_{lm\omega p})_{nmm}$	Δ	$1/r$	1	$1/r$
$(\eth_- C_{lm\omega p})_{l(nm)}$	1	$1/r^2$	1	$1/r^2$
$(\eth_- C_{lm\omega p})_{lmm}$	$1/\Delta$	1	1	$1/r^2$

they are given by

$$ {}_{\eth_+ g_{n\bar{m}}}S^{\rm in}_{lm\omega p} = \frac{4ik_{m\omega}\sqrt{Mr_+}\,{}_{-2}\kappa_{m\omega}}{\zeta_+}\mathscr{L}_{2(-m)(-\omega)}\,{}_{-2}\Theta_{lm\omega}, \tag{3.198a}$$

$$ {}_{\eth_+ g_{n\bar{m}}}S^{\rm out}_{lm\omega p} = \frac{{}_{-2}\eta^{+}_{lm\omega}\zeta_+ + 8Mr_+ik_{m\omega}\,{}_{-1}\kappa_{m\omega}}{4(Mr_+)^{3/2}ik_{m\omega}\,{}_{-1}\kappa_{m\omega}\zeta_+^2}\mathscr{L}_{2(-m)(-\omega)}\,{}_{-2}\Theta_{lm\omega}, \tag{3.198b}$$

$$ {}_{\eth_+ g_{n\bar{m}}}S^{\rm down}_{lm\omega p} = 2\sqrt{2}i\omega\mathscr{L}_{2(-m)(-\omega)}\,{}_{-2}\Theta_{lm\omega}, \tag{3.198c}$$

$$ {}_{\eth_+ g_{n\bar{m}}}S^{\rm up}_{lm\omega p} = -\sqrt{2}\mathscr{L}_{2(-m)(-\omega)}\,{}_{-2}\Theta_{lm\omega}, \tag{3.198d}$$

$$ {}_{\eth_+ g_{\bar{m}\bar{m}}}S^{\rm in}_{lm\omega p} = 4(2Mr_+)^{3/2}k^2_{m\omega}\,{}_{-2}\kappa_{m\omega}\,{}_{-1}\kappa_{m\omega}\,{}_{-2}\Theta_{lm\omega}, \tag{3.198e}$$

$$ {}_{\eth_+ g_{\bar{m}\bar{m}}}S^{\rm out}_{lm\omega p} = -\frac{24Mr_+i\omega k_{m\omega}\,{}_{-1}\kappa_{m\omega}\zeta_+ + [i\zeta_+(2-\,{}_{-1}\eta^{+}_{lm\omega}) + 8Mr_+k_{m\omega}]\,{}_{-2}\eta^{+}_{lm\omega}}{4ik^2_{m\omega}(2Mr_+)^{5/2}\,{}_{-1}\kappa_{m\omega}\zeta_+}\,{}_{-2}\Theta_{lm\omega}, \tag{3.198f}$$

$$ {}_{\eth_+ g_{\bar{m}\bar{m}}}S^{\rm down}_{lm\omega p} = 4\omega^2\,{}_{-2}\Theta_{lm\omega}, \tag{3.198g}$$

$$ {}_{\eth_+ g_{\bar{m}\bar{m}}}S^{\rm up}_{lm\omega p} = \frac{i\,{}_2\eta^{\infty}_{lm\omega}}{\omega}\,{}_{-2}\Theta_{lm\omega}, \tag{3.198h}$$

$$ {}_{\eth_- g_{lm}}S^{\rm in}_{lm\omega p} = \frac{{}_2\eta^{+}_{lm\omega}\zeta_+ - 8Mr_+ik_{m\omega}\,{}_1\kappa_{m\omega}}{8(Mr_+)^{3/2}ik_{m\omega}\,{}_1\kappa_{m\omega}}\mathscr{L}_{2m\omega}\,{}_2\Theta_{lm\omega}, \tag{3.198i}$$

$$ {}_{\eth_- g_{lm}}S^{\rm out}_{lm\omega p} = 2\sqrt{Mr_+}ik_{m\omega}\,{}_2\kappa_{m\omega}\zeta_+\mathscr{L}_{2m\omega}\,{}_2\Theta_{lm\omega}, \tag{3.198j}$$

$$ {}_{\eth_- g_{lm}}S^{\rm down}_{lm\omega p} = \frac{\mathscr{L}_{2m\omega}}{\sqrt{2}}\,{}_2\Theta_{lm\omega}, \tag{3.198k}$$

$$ {}_{\eth_- g_{lm}}S^{\rm up}_{lm\omega p} = \sqrt{2}i\omega\mathscr{L}_{2m\omega}\,{}_2\Theta_{lm\omega}, \tag{3.198l}$$

$$\mathfrak{d}_{-g_{mm}} S^{\text{in}}_{lm\omega p} = \frac{24Mr_+ i\omega k_{m\omega}\,{}_1\kappa_{m\omega}\zeta_+ + [i\zeta_+(2 - {}_1\eta^+_{lm\omega}) - 8Mr_+ k_{m\omega}]\,{}_2\eta^+_{lm\omega}}{16ik^2_{m\omega}(2Mr_+)^{5/2}\,{}_1\kappa_{m\omega}\zeta_+}\,{}_2\Theta_{lm\omega}, \tag{3.198m}$$

$$\mathfrak{d}_{-g_{mm}} S^{\text{out}}_{lm\omega p} = -(2Mr_+)^{3/2} k^2_{m\omega}\,{}_2\kappa_{m\omega}\,{}_1\kappa_{m\omega}\,{}_2\Theta_{lm\omega}, \tag{3.198n}$$

$$\mathfrak{d}_{-g_{mm}} S^{\text{down}}_{lm\omega p} = \frac{i\,{}_{-2}\eta^\infty_{lm\omega}}{4\omega}\,{}_2\Theta_{lm\omega}, \tag{3.198o}$$

$$\mathfrak{d}_{-g_{mm}} S^{\text{up}}_{lm\omega p} = -\omega^2\,{}_2\Theta_{lm\omega}, \tag{3.198p}$$

for the metric perturbation, and

$$\mathfrak{d}_{+C_{l\bar{m}\bar{m}}} S^{\text{in}}_{lm\omega p} = 4(2Mr_+)^{5/2} ik^3_{m\omega}\,{}_{-2}\kappa_{m\omega}\,{}_{-1}\kappa_{m\omega}\,{}_{-2}\Theta_{lm\omega}, \tag{3.199a}$$

$$\begin{aligned}\mathfrak{d}_{+C_{l\bar{m}\bar{m}}} S^{\text{out}}_{lm\omega p} = \Big\{&4Mr_+ ik_{m\omega}\,{}_1\kappa_{m\omega}[24Mr_+ i\omega k_{m\omega}\,{}_{-1}\kappa_{m\omega} + i(2 - {}_{-1}\eta^+_{lm\omega})\,{}_{-2}\eta^+_{lm\omega}] \\ &- \zeta_+\{8Mr_+ i\omega k_{m\omega}[3\,{}_{-1}\kappa_{m\omega}(2 - {}_1\eta^+_{lm\omega}) - 4\,{}_{-2}\eta^+_{lm\omega}] \\ &+ i\,{}_{-2}\eta^+_{lm\omega}[|{}_{-1}\eta^+_{lm\omega}|^2 + 4({}_2\lambda_{lm\omega} + 1)]\}\Big\}\frac{{}_{-2}\Theta_{lm\omega}}{16k^3_{m\omega}(2Mr_+)^{7/2}|{}_{-1}\kappa_{m\omega}|^2\zeta_+},\end{aligned} \tag{3.199b}$$

$$\mathfrak{d}_{+C_{l\bar{m}\bar{m}}} S^{\text{down}}_{lm\omega p} = 4i\omega^3\,{}_{-2}\Theta_{lm\omega}, \tag{3.199c}$$

$$\mathfrak{d}_{+C_{l\bar{m}\bar{m}}} S^{\text{up}}_{lm\omega p} = -\frac{i\,{}_2\eta^\infty_{lm\omega}}{2\omega}\,{}_{-2}\Theta_{lm\omega}, \tag{3.199d}$$

$$\mathfrak{d}_{+C_{n(l\bar{m})}} S^{\text{in}}_{lm\omega p} = -4(Mr_+)^{3/2} k^2_{m\omega}\,{}_{-2}\kappa_{m\omega}\,{}_{-1}\kappa_{m\omega}\zeta_+^{-2}(\zeta_+\mathscr{L}_{2(-m)(-\omega)} - ia\sin\theta)\,{}_{-2}\Theta_{lm\omega}, \tag{3.199e}$$

$$\begin{aligned}\mathfrak{d}_{+C_{n(l\bar{m})}} S^{\text{out}}_{lm\omega p} = \zeta_+^{-3}\Big\{&[24Mr_+ i\omega k_{m\omega}\,{}_{-1}\kappa_{m\omega} + i(2 - {}_{-1}\eta^+_{lm\omega})\,{}_{-2}\eta^+_{lm\omega}]\zeta_+(\zeta_+\mathscr{L}_{2(-m)(-\omega)} - ia\sin\theta) \\ &+ 8Mr_+ k_{m\omega}\,{}_{-2}\eta^+_{lm\omega}(2\zeta_+\mathscr{L}_{2(-m)(-\omega)} - ia\sin\theta) \\ &+ 6(4Mr_+)^2 ik^2_{m\omega}\,{}_{-1}\kappa_{m\omega}\mathscr{L}_{2(-m)(-\omega)}\Big\}\frac{{}_{-2}\Theta_{lm\omega}}{64(Mr_+)^{5/2} ik^2_{m\omega}\,{}_{-1}\kappa_{m\omega}},\end{aligned} \tag{3.199f}$$

$$\mathfrak{d}_{+C_{n(l\bar{m})}} S^{\text{down}}_{lm\omega p} = -\sqrt{2}\omega^2\mathscr{L}_{2(-m)(-\omega)}\,{}_{-2}\Theta_{lm\omega}, \tag{3.199g}$$

$$\begin{aligned}\mathfrak{d}_{+C_{n(l\bar{m})}} S^{\text{up}}_{lm\omega p} = -\Big\{&[4\omega^2 a^2(2\cos^2\theta - 3) - 12i\omega(M + iam) + {}_2\lambda_{lm\omega}({}_2\lambda_{lm\omega} + 2)]\mathscr{L}_{2(-m)(-\omega)} \\ &+ 4a\omega\sin\theta(12a\omega\cos\theta + {}_2\eta^\infty_{lm\omega})\Big\}\frac{{}_{-2}\Theta_{lm\omega}}{8\sqrt{2}\omega^2},\end{aligned} \tag{3.199h}$$

$$\mathfrak{d}_{+}C_{n\bar{m}\bar{m}}S^{\rm in}_{lm\omega p} = \sqrt{Mr_+}ik_{m\omega}\,{}_{-2}\kappa_{m\omega}\zeta_+^{-4}\Big\{\zeta_+^2[(2-\,{}_{-1}\eta^+_{lm\omega})-\mathscr{L}_{(-1)(-m)(-\omega)}\mathscr{L}_{2(-m)(-\omega)}] + 16Mr_+ik_{m\omega}\,{}_{-3/2}\kappa_{m\omega}\zeta_+ + 2a^2\sin^2\theta + ia\sin\theta\zeta_+\mathscr{L}_{2(-m)(-\omega)}\Big\}\frac{{}_{-2}\Theta_{lm\omega}}{\sqrt{2}}, \tag{3.199i}$$

$$\mathfrak{d}_{+}C_{n\bar{m}\bar{m}}S^{\rm out}_{lm\omega p} = \zeta_+^{-4}\Big\{\,{}_{-2}\eta^+_{lm\omega}[\zeta_+(ia\sin\theta-\zeta_+\mathscr{L}_{(-1)(-m)(-\omega)})\mathscr{L}_{2(-m)(-\omega)} - 8Mr_+ik_{m\omega}\zeta_+ + 2a^2\sin^2\theta] - 8Mr_+ik_{m\omega}\,{}_{-1}\kappa_{m\omega}(\zeta_+\mathscr{L}_{(-1)(-m)(-\omega)}-ia\sin\theta)\mathscr{L}_{2(-m)(-\omega)} + [24Mr_+\omega k_{m\omega}\,{}_{-1}\kappa_{m\omega}+(2-\,{}_{-1}\eta^+_{lm\omega})\,{}_{-2}\eta^+_{lm\omega}]\zeta_+^2\Big\}\frac{{}_{-2}\Theta_{lm\omega}}{(8Mr_+)^{3/2}ik_{m\omega}\,{}_{-1}\kappa_{m\omega}}, \tag{3.199j}$$

$$\mathfrak{d}_{+}C_{n\bar{m}\bar{m}}S^{\rm down}_{lm\omega p} = -5\omega^2\,{}_{-2}\Theta_{lm\omega}, \tag{3.199k}$$

$$\mathfrak{d}_{+}C_{n\bar{m}\bar{m}}S^{\rm up}_{lm\omega p} = -\frac{2\,{}_2\eta^\infty_{lm\omega}-\mathscr{L}_{(-1)(-m)(-\omega)}\mathscr{L}_{2(-m)(-\omega)}}{4}\,{}_{-2}\Theta_{lm\omega}, \tag{3.199l}$$

$$\mathfrak{d}_{-}C_{nmm}S^{\rm in}_{lm\omega p} = \Big\{4Mr_+ik_{m\omega}\,{}_{-1}\kappa_{m\omega}[24Mr_+i\omega k_{m\omega}\,{}_1\kappa_{m\omega}+i(2-\,{}_1\eta^+_{lm\omega})\,{}_2\eta^+_{lm\omega}] + \zeta_+\{8Mr_+i\omega k_{m\omega}[3\,{}_1\kappa_{m\omega}(2-\,{}_{-1}\eta^+_{lm\omega})-4\,{}_2\eta^+_{lm\omega}] + i\,{}_2\eta^+_{lm\omega}[|\,{}_1\eta^+_{lm\omega}|^2+4(\,{}_2\lambda_{lm\omega}+1)]\}\Big\}\frac{{}_2\Theta_{lm\omega}}{k^3_{m\omega}(8Mr_+)^{7/2}|\,{}_1\kappa_{m\omega}|^2\zeta_+^3}, \tag{3.199m}$$

$$\mathfrak{d}_{-}C_{nmm}S^{\rm out}_{lm\omega p} = -\frac{(2Mr_+)^{5/2}ik^3_{m\omega}\,{}_2\kappa_{m\omega}\,{}_1\kappa_{m\omega}}{2\zeta_+^2}\,{}_2\Theta_{lm\omega}, \tag{3.199n}$$

$$\mathfrak{d}_{-}C_{nmm}S^{\rm down}_{lm\omega p} = \frac{i\,{}_{-2}\eta^\infty_{lm\omega}}{16\omega}\,{}_2\Theta_{lm\omega}, \tag{3.199o}$$

$$\mathfrak{d}_{-}C_{nmm}S^{\rm up}_{lm\omega p} = -\frac{i\omega^3}{2}\,{}_2\Theta_{lm\omega}, \tag{3.199p}$$

$$\mathfrak{d}_{-}C_{l(nm)}S^{\rm in}_{lm\omega p} = \zeta_+^{-3}\Big\{[24Mr_+i\omega k_{m\omega}\,{}_1\kappa_{m\omega}+i\,{}_2\eta^+_{lm\omega}(2-\,{}_1\eta^+_{lm\omega})]\zeta_+(\zeta_+\mathscr{L}_{2m\omega}+ia\sin\theta) - 8Mr_+k_{m\omega}\,{}_2\eta^+_{lm\omega}(2\zeta_+\mathscr{L}_{2m\omega}+ia\sin\theta) + 6(4Mr_+)^2ik^2_{m\omega}\,{}_1\kappa_{m\omega}\mathscr{L}_{2m\omega}\Big\}\frac{{}_2\Theta_{lm\omega}}{256(Mr_+)^{5/2}ik^2_{m\omega}\,{}_1\kappa_{m\omega}}, \tag{3.199q}$$

$$\mathfrak{d}_{-}C_{l(nm)}S^{\rm out}_{lm\omega p} = -(Mr_+)^{3/2}k^2_{m\omega}\,{}_2\kappa_{m\omega}\,{}_1\kappa_{m\omega}\zeta_+^{-2}(\zeta_+\mathscr{L}_{2m\omega}+ia\sin\theta)\,{}_2\Theta_{lm\omega}, \tag{3.199r}$$

$$\mathfrak{d}_{-}C_{l(nm)}S^{\rm down}_{lm\omega p} = -\Big\{[4\omega^2a^2(2\cos^2\theta-3)+12i\omega(M-iam)+\,{}_2\lambda_{lm\omega}(\,{}_2\lambda_{lm\omega}+2)]\mathscr{L}_{2m\omega} + 4a\omega\sin\theta(12a\omega\cos\theta+\,{}_{-2}\eta^\infty_{lm\omega})\Big\}\frac{{}_2\Theta_{lm\omega}}{32\sqrt{2}\omega^2}, \tag{3.199s}$$

$$\mathfrak{d}_{-}C_{l(nm)}S^{\rm up}_{lm\omega p} = -\frac{\omega^2\mathscr{L}_{2m\omega}}{2\sqrt{2}}\,{}_2\Theta_{lm\omega}, \tag{3.199t}$$

$$\delta_{-}C_{lmm}S^{\text{in}}_{lm\omega p} = \zeta^{-2}\Big\{ {}_2\eta^{+}_{lm\omega}[2a^2\sin^2\theta - \zeta_+(\zeta_+\mathscr{L}_{(-1)m\omega} + 3ia\sin\theta)\mathscr{L}_{2m\omega} - 8Mr_+ik_{m\omega}\zeta_+] + 8Mr_+ik_{m\omega}\,{}_1\kappa_{m\omega}(\zeta_+\mathscr{L}_{(-1)m\omega} + 3ia\sin\theta)\mathscr{L}_{2m\omega} - [24Mr_+\omega k_{m\omega}\,{}_1\kappa_{m\omega} + {}_2\eta^{+}_{lm\omega}(2 - {}_1\eta^{+}_{lm\omega})]\zeta_+^2\Big\}\frac{{}_2\Theta_{lm\omega}}{2(8Mr_+)^{3/2}ik_{m\omega}\,{}_1\kappa_{m\omega}}, \tag{3.199u}$$

$$\delta_{-}C_{lmm}S^{\text{out}}_{lm\omega p} = \sqrt{Mr_+}ik_{m\omega}\,{}_2\kappa_{m\omega}\zeta_+^{-2}\Big\{\zeta_+(8Mr_+ik_{m\omega}\,{}_2\kappa_{m\omega} - 3ia\sin\theta\mathscr{L}_{2m\omega}) + 2a^2\sin^2\theta - \zeta_+^2[\mathscr{L}_{(-1)m\omega}\mathscr{L}_{2m\omega} + (2 - {}_1\eta^{+}_{lm\omega})]\Big\}\frac{{}_2\Theta_{lm\omega}}{2\sqrt{2}}, \tag{3.199v}$$

$$\delta_{-}C_{lmm}S^{\text{down}}_{lm\omega p} = -\frac{2\,{}_{-2}\eta^{\infty}_{lm\omega} + \mathscr{L}_{(-1)m\omega}\mathscr{L}_{2m\omega}}{8}\,{}_2\Theta_{lm\omega}, \tag{3.199w}$$

$$\delta_{-}C_{lmm}S^{\text{up}}_{lm\omega p} = -\frac{3\omega^2}{2}\,{}_2\Theta_{lm\omega}, \tag{3.199x}$$

for the perturbed connection coefficients, where $\zeta_+ = r_+ + ia\cos\theta$.

3.5.3 Results

At this point, we can use the results of sections 3.5.1 and 3.5.2 to compute our final results. We begin with the fluxes of the currents ${}_s\mathcal{C}j^a\{\mathrm{Im}[\delta_+\boldsymbol{g}]\}$ and ${}_s\mathcal{D}j^a\{\mathrm{Im}[\delta_+\boldsymbol{g}]\}$. In terms of the fluxes of the currents (3.188), we have that (averaging over w and α)

$$\left\langle \frac{\mathrm{d}^2\,{}_s\mathcal{C}Q\{\mathrm{Im}[\delta_+\boldsymbol{g}]\}}{\mathrm{d}w\mathrm{d}\Omega}\right\rangle_{w,\alpha} = \frac{i}{32}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l,l'=2}^{\infty}\sum_{|m|\leq\min(l,l')}\sum_{p,p'=\pm1}pp'\,{}_sC_{lm\omega p}\overline{{}_sC_{l'm\omega p'}}\frac{\mathrm{d}^2\,{}_sQ_{ll'm\omega pp'}}{\mathrm{d}w\mathrm{d}\Omega}, \tag{3.200a}$$

$$\left\langle \frac{\mathrm{d}^2\,{}_s\mathcal{D}Q\{\mathrm{Im}[\delta_+\boldsymbol{g}]\}}{\mathrm{d}w\mathrm{d}\Omega}\right\rangle_{w,\alpha} = \frac{i}{16}\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{l,l'=2}^{\infty}\sum_{|m|\leq\min(l,l')}\sum_{p,p'=\pm1}{}_2\lambda_{l'm\omega}\frac{\mathrm{d}^2\,{}_sQ_{ll'm\omega pp'}}{\mathrm{d}w\mathrm{d}\Omega}. \tag{3.200b}$$

As these fluxes are all $\mathbb{R}$-bilinear, it is convenient to define

$${}_s\Upsilon^{\text{in/out/down/up}}_{ll'm\omega pp'} \equiv {}_s\widehat{\psi}^{\text{in/out/down/up}}_{lm\omega p}\overline{{}_s\widehat{\psi}^{\text{in/out/down/up}}_{l'm\omega p'}}. \tag{3.201}$$

Using table 3.2 and equations (3.189a) and (3.189b), we find that

$$\left.\frac{\mathrm{d}^2\,{}_{+2}Q^{\text{down}}_{ll'm\omega pp'}}{\mathrm{d}u\mathrm{d}\Omega}\right|_{\mathscr{I}+} = 0, \tag{3.202a}$$

$$\left.\frac{\mathrm{d}^2\,{}_{+2}Q^{\text{down}}_{ll'm\omega pp'}}{\mathrm{d}v\mathrm{d}\Omega}\right|_{\mathscr{I}-} = -\frac{i}{64\pi}\,{}_{-2}\Upsilon^{\text{down}}_{ll'm\omega pp'}\,{}_{\eth_+}C_{l\bar{m}\bar{m}}\,S^{\text{down}}_{lm\omega p}\,\overline{{}_{\eth_+}g_{\bar{m}\bar{m}}\,S^{\text{down}}_{l'm\omega p'}} + \overline{l,p \longleftrightarrow l',p'}, \tag{3.202b}$$

$$\left.\frac{\mathrm{d}^2\,{}_{+2}Q^{\text{down}}_{ll'm\omega pp'}}{\mathrm{d}v\mathrm{d}\Omega}\right|_{H+} = -\frac{i}{64\pi}\,{}_{-2}\Upsilon^{\text{in}}_{ll'm\omega pp'}\,{}_{\eth_+}C_{l\bar{m}\bar{m}}\,S^{\text{in}}_{lm\omega p}\,\overline{{}_{(\eth_+ g)_{\bar{m}\bar{m}}}\,S^{\text{in}}_{l'm\omega p'}} + \overline{l,p \longleftrightarrow l',p'}, \tag{3.202c}$$

$$\left.\frac{\mathrm{d}^2\,{}_{+2}Q^{\text{down}}_{ll'm\omega pp'}}{\mathrm{d}u\mathrm{d}\Omega}\right|_{H-} = \frac{i\Sigma_+}{32\pi}\,{}_{-2}\Upsilon^{\text{out}}_{ll'm\omega pp'}\left({}_{\eth_+}C_{n\bar{m}\bar{m}}\,S^{\text{out}}_{lm\omega p}\,\overline{{}_{\eth_+}g_{\bar{m}\bar{m}}\,S^{\text{out}}_{l'm\omega p'}} - {}_{\eth_+}C_{n(l\bar{m})}\,S^{\text{out}}_{lm\omega p}\,\overline{{}_{\eth_+}g_{n\bar{m}}\,S^{\text{out}}_{l'm\omega p'}}\right)$$
$$+ \overline{l,p \longleftrightarrow l',p'}, \tag{3.202d}$$

where $\Sigma_+ = |\zeta_+|^2$, and the superscript "down" indicates that we have performed a projection such that ${}_s\widehat{\psi}^{\text{up}}_{lm\omega p} = 0$. Similarly,

$$\left.\frac{\mathrm{d}^2\,{}_{-2}Q^{\text{up}}_{ll'm\omega pp'}}{\mathrm{d}u\mathrm{d}\Omega}\right|_{\mathscr{I}+} = \frac{i}{32\pi}\,{}_{2}\Upsilon^{\text{up}}_{ll'm\omega pp'}\,{}_{\eth_-}C_{nmm}\,S^{\text{up}}_{lm\omega p}\,\overline{{}_{\eth_-}g_{mm}\,S^{\text{up}}_{l'm\omega p'}} + \overline{l,p \longleftrightarrow l',p'}, \tag{3.203a}$$

$$\left.\frac{\mathrm{d}^2\,{}_{-2}Q^{\text{up}}_{ll'm\omega pp'}}{\mathrm{d}v\mathrm{d}\Omega}\right|_{\mathscr{I}-} = 0, \tag{3.203b}$$

$$\left.\frac{\mathrm{d}^2\,{}_{-2}Q^{\text{up}}_{ll'm\omega pp'}}{\mathrm{d}v\mathrm{d}\Omega}\right|_{H+} = -\frac{i}{64\pi}\,{}_{2}\Upsilon^{\text{in}}_{ll'm\omega pp'}\left({}_{\eth_-}C_{lmm}\,S^{\text{in}}_{lm\omega p}\,\overline{{}_{\eth_-}g_{mm}\,S^{\text{in}}_{l'm\omega p'}} - {}_{\eth_-}C_{l(nm)}\,S^{\text{in}}_{lm\omega p}\,\overline{{}_{\eth_-}g_{lm}\,S^{\text{in}}_{l'm\omega p'}}\right)$$
$$+ \overline{l,p \longleftrightarrow l',p'}, \tag{3.203c}$$

$$\left.\frac{\mathrm{d}^2\,{}_{-2}Q^{\text{up}}_{ll'm\omega pp'}}{\mathrm{d}u\mathrm{d}\Omega}\right|_{H-} = \frac{i\Sigma_+}{32\pi}\,{}_{2}\Upsilon^{\text{out}}_{ll'm\omega pp'}\,{}_{\eth_-}C_{nmm}\,S^{\text{out}}_{lm\omega p}\,\overline{{}_{\eth_-}g_{mm}\,S^{\text{out}}_{l'm\omega p'}} + \overline{l,p \longleftrightarrow l',p'}, \tag{3.203d}$$

and the superscript "up" denotes the fact that we have performed a projection to set ${}_s\widehat{\psi}^{\text{down}}_{lm\omega p} = 0$. If these projections are not performed, then the respective fluxes *diverge*, as is evident from table 3.2 and equation (3.189). Since the fluxes of ${}_{s\mathcal{C}}j^a$ and ${}_{s\mathcal{D}}j^a$ can be written in terms of those of ${}_sj^a_{ll'm\omega pp'}$, there are issues with these currents as well.

These divergences motivated the introduction of the projection operators in section 3.2.5. With these projection operators, we have sacrificed locality (which we had already sacrificed in ${}_{s\mathcal{D}}j^a$) in order to obtain finite fluxes. As mentioned at the end of section 3.4.3, their geometric optics limits are similar to those of the currents ${}_s\mathcal{C}j^a$ and ${}_s\mathcal{D}j^a$, as presented in table 3.1. We also have that

$$\left\langle \frac{\mathrm{d}^2\ {}_2\mathring{\mathcal{C}}Q\{\mathrm{Im}[\mathring{\eth}_+\boldsymbol{g}]\}}{\mathrm{d}w\mathrm{d}\Omega} \right\rangle_{w,\alpha} = \frac{1}{32i}\int_{-\infty}^{\infty}\mathrm{d}\omega \sum_{l,l'=2}^{\infty}\sum_{|m|\leq\min(l,l')}\sum_{p,p'=\pm1} \times pp'\left\{ {}_2C_{lm\omega p}\overline{{}_2C_{l'm\omega p'}}\frac{\mathrm{d}^2\ {}_2Q^{\mathrm{down}}_{ll'm\omega pp'}}{\mathrm{d}w\mathrm{d}\Omega} + {}_{-2}C_{lm\omega p}\overline{{}_{-2}C_{l'm\omega p'}}\frac{\mathrm{d}^2\ {}_{-2}Q^{\mathrm{up}}_{ll'm\omega pp'}}{\mathrm{d}w\mathrm{d}\Omega}\right\}, \tag{3.204a}$$

$$\left\langle \frac{\mathrm{d}^2\ {}_2\mathring{\mathcal{D}}Q\{\mathrm{Im}[\mathring{\eth}_+\boldsymbol{g}]\}}{\mathrm{d}w\mathrm{d}\Omega} \right\rangle_{w,\alpha} = \frac{1}{16i}\int_{-\infty}^{\infty}\mathrm{d}\omega \sum_{l,l'=2}^{\infty}\sum_{|m|\leq\min(l,l')}\sum_{p,p'=\pm1} \times {}_2\lambda_{l'm\omega}\left\{\frac{\mathrm{d}^2\ {}_2Q^{\mathrm{down}}_{ll'm\omega pp'}}{\mathrm{d}w\mathrm{d}\Omega} + \frac{\mathrm{d}^2\ {}_{-2}Q^{\mathrm{up}}_{ll'm\omega pp'}}{\mathrm{d}w\mathrm{d}\Omega}\right\}. \tag{3.204b}$$

Using equations (3.202), (3.203), and (3.204), we have completely determined the fluxes of the charges associated with ${}_2\mathring{\mathcal{C}}j^a\{\mathrm{Im}[\mathring{\eth}_+\boldsymbol{g}]\}$ and ${}_2\mathring{\mathcal{D}}j^a\{\mathrm{Im}[\mathring{\eth}_+\boldsymbol{g}]\}$.

Using the symplectic product for linearized gravity, we have not been able to construct a *local* current with finite fluxes which reduces to the Carter constant in geometric optics. However, we can do so using the symplectic product we defined in equation (3.143) for the master variables. We find that the fluxes for ${}_s\Omega j^a[\delta\boldsymbol{g}]$, averaged over w and α, are given by an expansion of the form

$$\left\langle \frac{\mathrm{d}^2\ {}_s\Omega Q[\delta\boldsymbol{g}]}{\mathrm{d}w\mathrm{d}\Omega} \right\rangle_{w,\alpha} \equiv \int_{-\infty}^{\infty}\mathrm{d}\omega \sum_{l,l'=2}^{\infty}\sum_{|m|<l,l'}\sum_{p,p'=\pm1}\frac{\mathrm{d}^2\ {}_s\Omega Q_{ll'm\omega pp'}}{\mathrm{d}w\mathrm{d}\Omega}, \tag{3.205}$$

where

$$\left.\frac{\mathrm{d}^2\ {}_s\Omega Q_{ll'm\omega pp'}}{\mathrm{d}u\mathrm{d}\Omega}\right|_{\mathscr{I}+} = \frac{\omega}{32\pi}\left\{C_{l'm\omega}\,{}_s\Theta_{lm\omega}\,{}_s\Theta_{l'm\omega}\left[{}_s\widehat{\psi}^{\mathrm{up}}_{lm\omega p}\overline{{}_{-s}\widehat{\psi}^{\mathrm{up}}_{l'm\omega p'}} + \overline{l,p,s\longleftrightarrow l',p',-s}\right] + {}_sC_{l'm\omega p'}\,{}_{-s}\Theta_{lm\omega}\,{}_{-s}\Theta_{l'm\omega}\left[{}_{-s}\widehat{\psi}^{\mathrm{up}}_{lm\omega p}\overline{{}_s\widehat{\psi}^{\mathrm{up}}_{l'm\omega p'}} + \overline{l,p,s\longleftrightarrow l',p',-s}\right]\right\}, \tag{3.206a}$$

$$\left.\frac{\mathrm{d}^2\ {}_s\Omega Q_{ll'm\omega pp'}}{\mathrm{d}v\mathrm{d}\Omega}\right|_{\mathscr{I}-} = -\frac{\omega}{32\pi}\left\{C_{l'm\omega}\,{}_s\Theta_{lm\omega}\,{}_s\Theta_{l'm\omega}\left[{}_s\widehat{\psi}^{\mathrm{down}}_{lm\omega p}\overline{{}_{-s}\widehat{\psi}^{\mathrm{down}}_{l'm\omega p'}} + \overline{l,p,s\longleftrightarrow l',p',-s}\right] + {}_sC_{l'm\omega p'}\,{}_{-s}\Theta_{lm\omega}\,{}_{-s}\Theta_{l'm\omega}\left[{}_{-s}\widehat{\psi}^{\mathrm{down}}_{lm\omega p}\overline{{}_s\widehat{\psi}^{\mathrm{down}}_{l'm\omega p'}} + \overline{l,p,s\longleftrightarrow l',p',-s}\right]\right\}, \tag{3.206b}$$

and

$$\left.\frac{\mathrm{d}^2\,{}_s\Omega Q_{ll'm\omega pp'}}{\mathrm{d}v\mathrm{d}\Omega}\right|_{H^+} = -\frac{Mr_+k_{m\omega}}{16\pi}\Bigg\{C_{l'm\omega}\,{}_s\Theta_{lm\omega}\,{}_s\Theta_{l'm\omega}\Big[{}_s\kappa_{m\omega}\,{}_s\widehat{\psi}^{\mathrm{in}}_{lm\omega p}\,\overline{{}_{-s}\widehat{\psi}^{\mathrm{in}}_{l'm\omega p'}} + \overline{l,p,s\longleftrightarrow l',p',-s}\Big] + {}_sC_{l'm\omega p'}\,{}_{-s}\Theta_{lm\omega}\,{}_{-s}\Theta_{l'm\omega}\Big[{}_{-s}\kappa_{m\omega}\,{}_{-s}\widehat{\psi}^{\mathrm{in}}_{lm\omega p}\,\overline{{}_s\widehat{\psi}^{\mathrm{in}}_{l'm\omega p'}} + \overline{l,p,s\longleftrightarrow l',p',-s}\Big]\Bigg\}, \tag{3.207a}$$

$$\left.\frac{\mathrm{d}^2\,{}_s\Omega Q_{ll'm\omega pp'}}{\mathrm{d}u\mathrm{d}\Omega}\right|_{H^-} = \frac{Mr_+k_{m\omega}}{16\pi}\Bigg\{C_{l'm\omega}\,{}_s\Theta_{lm\omega}\,{}_s\Theta_{l'm\omega}\Big[{}_s\kappa_{m\omega}\,{}_s\widehat{\psi}^{\mathrm{out}}_{lm\omega p}\,\overline{{}_{-s}\widehat{\psi}^{\mathrm{out}}_{l'm\omega p'}} + \overline{l,p,s\longleftrightarrow l',p',-s}\Big] + {}_sC_{l'm\omega p'}\,{}_{-s}\Theta_{lm\omega}\,{}_{-s}\Theta_{l'm\omega}\Big[{}_{-s}\kappa_{m\omega}\,{}_{-s}\widehat{\psi}^{\mathrm{out}}_{lm\omega p}\,\overline{{}_s\widehat{\psi}^{\mathrm{out}}_{l'm\omega p'}} + \overline{l,p,s\longleftrightarrow l',p',-s}\Big]\Bigg\}, \tag{3.207b}$$

where $\overline{l,p,s\longleftrightarrow l',p',-s}$ means "the complex conjugate of the previous terms, with l, p, and s switched with l', p', and $-s$". Note that, by equations (3.124), it is possible to write these expressions entirely in terms of ${}_s\Upsilon^{\mathrm{in/out/down/up}}_{ll'm\omega pp'}$, although we do not do so for brevity.

3.6 Discussion

In this chapter, we have constructed a class of conserved currents for linearized gravity whose conserved charges reduce to the sum of the Carter constants (to some positive power) for a null fluid of gravitons in the geometric optics limit. These conserved currents are constructed from symplectic products of two solutions constructed via the method of symmetry operators. Moreover, some of these currents yield finite fluxes at the horizon and null infinity, although most that are finite at null infinity are not local.

That some of these currents possess diverging fluxes at null infinity is not ideal. It may be possible to find a symmetry operator, differing from those that appear in this chapter by a gauge transformation, that is both local and maps to a solution with a non-divergent symplectic product. In the absence of a clear example of such a symmetry operator, we have instead decided to consider nonlocal symmetry operators which are easier to define. We have also shown that there exists a symplectic product for the master variables (instead of the metric perturbation) which yields finite fluxes. This symplectic product can also be used to construct a current which gives (positive powers of) the Carter constant in the limit of geometric optics. However, note that this is not the physical symplectic product for linearized gravity.

One motivation for seeking conserved currents is the hope to derive, for the dynamical system of a point particle coupled to linearized gravity in the Kerr spacetime, a “unified conservation law” that would generalize the conservation of the Carter constant for a point particle by itself. The local currents considered in this chapter could be relevant for such a conservation law, but the potential relevance of the nonlocal currents is less obvious. We plan to further explore these currents, particularly their applications, in future work.

Chapter 4

Persistent Observables and the Measurement of Angular Momentum

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We now move to a different area of research, that of *persistent gravitational wave observables*. These observables are generalizations of the gravitational wave memory effect [187], which is the permanent change in separation of two closely-separated observers after a burst of gravitational waves. These generalizations share with this effect the following two properties: they are nonlocal in time (being measured over a time interval) and are vanishing if no gravitational waves were present during the course of this measurement.

In the present chapter, I will review three main examples of persistent observables which my collaborators and I introduced in [67]. The first is a straightforward generalization of the gravitational wave memory, which is called *curve deviation*. To measure this observable, two closely-separated observers measure their separation and relative velocity at early times, and then measure their separation again at late times. Moreover, they measure their proper accelerations at all times using accelerometers. Using the initial separation, initial relative velocity, and the accelerometer measurements, they can predict the final separation under the assumption that spacetime is flat. The difference between the final separation and this prediction is the curve deviation observable.

The second observable is a holonomy measurement, that is, a comparison of the initial and final values of the solution to an ordinary differential equation solved around a closed loop in spacetime. The particular differential equation in question is inspired by the following idea: the measured angular momentum of a particle in flat space changes depending upon the origin about which it is measured, and if this origin is varied along a curve, then the linear and angular momentum satisfy a coupled ordinary differential equation along this curve. Making this differential equation covariant, one obtains a set of coupled differential equations that one can solve around loops in spacetime.

Finally, the last observable is another direct generalization of the gravitational wave memory, where now one of the observers is replaced with a spinning test particle. The non-spinning observer can now measure not only the initial and final separations, but also the initial and final linear momenta and spins of the spinning particle. This observable is a sort of hybrid between the holonomy and curve deviation observables, as the intrinsic spin and momentum obey ordinary differential equations along the worldline of the spinning particle, but the worldline of the spinning particle is itself accelerated due to the presence of spin.

The structure of this chapter is as follows. In section 4.1, we review the definition of persistent observables, and then provide examples of persistent observables from the literature. We moreover introduce the persistent observables discussed above which were defined in [67], and briefly discuss the feasibility of their measurement. In section 4.2, we then turn to the computation of these observables in arbitrary spacetimes that are initially flat, have a burst of gravitational waves, and then settle down to being flat once again. This section (as well as this chapter generally) heavily relies on the theory of *covariant bitensors*, tensor fields which depend on multiple points in a spacetime, which we review in section 4.2.1; those who are unfamiliar with bitensors are encouraged to read this section first, as it also contains a variety of examples of the use of bitensors. Finally, in section 4.3, we consider persistent observables in a specific example of plane wave spacetimes. These spacetimes provide a simple context in which to study the nonlinear properties of persistent observables, as demonstrated in other work discussing persistent observables in the literature (see, for example, [189, 188, 190, 89]).

The conventions and notation particular to this chapter are the following. We use the conventions for taking the duals of arbitrary tensors from Penrose and Rindler [129, 130] (reviewed in appendix 4.A), and the conventions for bitensors from Poisson's review article [131], which we review in section 4.2.1. We will use capital Latin indices for tensor indices on an arbitrary vector bundle. For brevity, we are using a convention for bitensors where we use the same annotations for indices as are used on the points at which the indices apply (e.g., a, b at the point x and a', b' at the point x'). If a bitensor is a scalar at some point, we make the dependence on that point explicit. Finally, for brevity, we will occasionally take powers of order symbols, writing (for example) $O(a, b)^3$ as shorthand for $O(a^3, a^2 b, ab^2, b^3)$.

4.1 Persistent Observables

A *persistent gravitational wave observable*, or simply "persistent observable", is a measurement which a set of observers can perform during some time interval which always vanishes in the absence of gravitational waves during the given time interval. A special subclass of these observables are the *memory observables*, which are persistent observables that are associated with symmetries at spacetime boundaries, such as null infinity (as in the initial investigations of the memory effect [187, 53]) or the event horizon of a black hole [91, 57, 50]. Memory observables are some of the most commonly considered persistent observable (see, for example, [156] and references therein). However, the notion of a persistent observable is more general than that of a memory observable, and can be considered in the interior of a spacetime. For example, they are often considered in exact gravitational plane wave spacetimes (see, for example, [89, 189]).

The definition of a persistent observable given above is intentionally vague, since there is no universal definition of "nonradiative" regions of a spacetime in which these observables would vanish. Definitions of "nonradiative" exist in particular contexts, such as near null infinity or in linearized gravity with a fixed background. Therefore, a persistent observable is really a collection of several pieces:

1. a context in which "nonradiative" is well-defined,

2. a set of observers whose properties can be specified unambiguously, and

3. a measurement that these observers can perform over a given time interval that vanishes if there is no gravitational radiation during this interval.

An example of a persistent observable, in this language, can be given as follows: the context "linearized gravity on a flat background", the set of "two freely falling observers that are initially comoving", and the measurement of "the change in separation of the two observers over the time interval". One minor detail to keep in mind is that the measurement procedure for a given persistent observable, when measured in a different context, may not actually be part of a persistent observable. An example of this is given by the above example: consider the same observers and measurement procedure, but the context "linearized gravity on a fixed Schwarzschild background". The measurement would no longer vanish in the absence of radiation, since two initially comoving observers in Schwarzschild, generically, will either approach or separate from one another due to tidal forces.

In this chapter, we will (for simplicity) only consider the context where the spacetime is composed of three regions: an early, flat region I, a curved region II, and a late time, flat region III. Here "nonradiative" means "region II is flat". Persistent observables in this context are generally equivalent to integrated measures of curvature. As such, to leading order in the Riemann tensor, these observables measure moments (in time) of the Riemann tensor and its derivatives along an observer's worldline.

The observables in this section are all defined in a context where there are two flat regions of spacetime separated in time by a region with curvature. One can also consider situations where there are two flat regions that are spatially separated, as occurs (for example) when considering gravitational lensing. Here, there are effects of the intervening curvature on the propagation of null rays from sources to observers in astronomical observations. In this context, a number of nonlocal observables can be defined (related to lensing, frequency shifts, etc.; see, for example, [89, 84, 29, 95]) which bear some similarities to the observables discussed here.

In the remainder of this section, we will review previous examples of persistent gravitational wave observables in the literature in section 4.1.1. We then define our three new observables in sections 4.1.2, 4.1.3, and 4.1.4. For convenience, we give expressions in this section for the values of these persistent observables, both old and new, in a weak curvature limit; these are a simplified version of our results, and can be derived using the full results in section 4.2.3. Finally, we give a brief discussion as to the feasibility of the measurement of these persistent observables in 4.1.5. A summary of the different persistent observables that occur in this section is given in table 4.1.

As a brief note on notation, in this section we denote by lowercase Greek letters the components of tensors with respect to a basis that is parallel-transported along a given curve γ. That is, for a tensor $Q^{a_1\cdots a_r}{}_{b_1\cdots b_s}$, we write

$$Q^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s}(\tau) \equiv Q^{a_1\cdots a_r}{}_{b_1\cdots b_s}(\omega^{\alpha_1})_{a_1}\cdots(\omega^{\alpha_r})_{a_r}(e_{\beta_1})^{b_1}\cdots(e_{\beta_s})^{b_s}, \tag{4.1}$$

where $\{(e_\alpha)^a \mid \alpha = 0, \cdots, 3\}$ is a parallel-transported basis and $\{(\omega^\alpha)_a \mid \alpha = 0, \cdots, 3\}$ the corresponding parallel-transported dual basis. All tensors on the right-hand side of this equation are being evaluated at $\gamma(\tau)$; when the corresponding scalar on the left-hand side is independent of τ (for example, the tangent vector $\dot{\gamma}^\alpha$, if γ is a geodesic affinely parametrized by τ), the dependence on τ on the left-hand side is omitted. For brevity, we follow the usual Einstein summation convention for these basis components.

4.1.1 Examples of persistent observables from the literature

The prototypical example of a persistent gravitational observable is the *displacement memory observable*: the difference between the separation of two initially comoving, freely falling, and closely-separated observers measured before and after a burst of gravitational waves. Denote the curves that the two observers follow by γ and $\bar{\gamma}$, and the initial and final proper times of one of the observers by τ_0 and τ_1. At each of two points $\gamma(\tau_0)$ and $\gamma(\tau_1)$, if $\bar{\gamma}$ is close enough, there is a unique geodesic that intersects both γ and $\bar{\gamma}$ and is orthogonal to γ. We define the affine parameter λ along these unique connecting geodesics such that they intersect γ at $\lambda = 0$ and $\bar{\gamma}$ at $\lambda = 1$. The initial and final separation vectors ξ^a and $\xi^{a'}$ are then the tangent vectors to these unique connecting geodesics at $\gamma(\tau_0)$ and $\gamma(\tau_1)$, respectively.

This change in the separation is defined by taking the difference between the components of ξ^a and $\xi^{a'}$ on a basis parallel-transported along γ, and this change can then be found explicitly by solving the geodesic deviation equation. For initially comoving observers, this change is given by

$$\begin{aligned}\Delta\xi^\mu(\tau_1,\tau_0) &\equiv \xi^\mu(\tau_1) - \xi^\mu(\tau_0) \\ &= -\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_0}^{\tau_2} \mathrm{d}\tau_3 R^\mu{}_{\alpha\nu\beta}(\tau_3)\dot{\gamma}^\alpha\dot{\gamma}^\beta\xi^\nu(\tau_0) + O(\boldsymbol{\xi}^2, \boldsymbol{R}^2).\end{aligned} \tag{4.2}$$

Table 4.1: A summary of the persistent observables discussed in this section. We provide the original reference for the observable (if it was defined before [67]). As a brief summary of the characteristics of these observables, we also give the number of time integrals of the Riemann tensor which appear in these observables (in the weak curvature and plane-wave limits; see section 4.1.5 for more details) and the known scaling near null infinity in both linearized gravity and in full general relativity. If the observable is known to be associated with some symmetry near a spacetime boundary (and so is a memory observable), that is indicated in the last column.

Observable	Number of time integrals of the Riemann tensor	Scaling near $\mathscr{I}$ (if known)		Associated with a known symmetry
		Linearized GR	Full GR	
Displacement [187]	2	$1/r$	$1/r$	Supertranslations [157]
Relative velocity [86]	1	$1/r^2$	$\cdots$	No
Relative rotation [68]	1	$1/r^2$	$\cdots$	No
Relative proper time [157]	1	$1/r^2$	$\cdots$	No
Subleading displacement[a] [127, 122]	3	$1/r$	$1/r$	Superrotations [127, 72, 122]
Curve deviation	1–3[b]	$\cdots$	$\cdots$	No
Holonomy	1–3[b]	$\cdots$	$\cdots$	No
Spinning test particle	1–2	$\cdots$	$\cdots$	No

[a]Subleading displacement memory near null infinity includes the spin memory [127] and center of mass memory [122].
[b]With acceleration, the number of time integrals is 4 and higher.

There is another type of displacement memory observable, which depends instead on the initial relative velocity $\dot{\xi}^\alpha$ of the two worldlines. This observable is the final relative displacement of two observers with no initial relative displacement, but an initial relative velocity. An explicit expression, defined and derived in a manner similar to equation (4.2), is given by

$$\widetilde{\Delta\xi}^\mu(\tau_1,\tau_0) = -\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_0}^{\tau_2} \mathrm{d}\tau_3 \int_{\tau_3}^{\tau_2} \mathrm{d}\tau_4 R^\mu{}_{\alpha\nu\beta}(\tau_4)\dot{\gamma}^\alpha\dot{\gamma}^\beta\dot{\xi}^\nu(\tau_0) + O(\boldsymbol{\xi}^2, \boldsymbol{R}^2). \tag{4.3}$$

This we will call the *subleading displacement memory observable.* It is called subleading because of the additional time integral in equation (4.3) relative to equation (4.2). The parts of the gravitational waves that produce the subleading displacement memory arise at a higher order in the post-Newtonian expansion than the parts that generate the leading memory [124, 122]. Subleading displacement memory has been studied exclusively at null infinity, where it has been understood in terms of its electric and magnetic parity components, which are known as *center of mass memory* [122] and *spin memory* [127], respectively. The total subleading displacement is a memory observable, since both the spin and the center of mass memories are known to be associated with superrotations, asymptotic symmetries which arise in an extension of the usual BMS group [28] (the spin memory is known to be associated with a soft theorem as well [127]).

Since the geodesic deviation equation has solutions where the initially comoving observers have four-velocities that become different over time, there is a *persistent relative velocity observable* given by a difference in the relative four-velocities before and after a burst of gravitational waves. This is often referred to as the velocity memory in the literature [86, 167, 89]. It takes the form

$$\begin{aligned}\Delta\dot{\xi}^{\mu}(\tau_1,\tau_0) &\equiv \partial_{\tau_1}\Delta\xi^{\mu}(\tau_1,\tau_0)\\ &= -\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 R^{\mu}{}_{\alpha\nu\beta}(\tau_2)\dot{\gamma}^{\alpha}\dot{\gamma}^{\beta}\xi^{\nu}(\tau_0) + O(\boldsymbol{\xi}^2,\boldsymbol{R}^2).\end{aligned} \tag{4.4}$$

The relative velocity observable is guaranteed to be nonvanishing in the context of nonlinear plane waves [36] (for a more recent discussion, see [189, 188, 190], as well as the discussion in section 4.3.1). It has been suggested, moreover, that it is in principle measurable for bursts generated by astrophysical sources [86]. Note that in the case where the final relative velocity is nonzero, the displacement memory will no longer be independent of the final time τ_1, even after the burst of gravitational waves has passed.

Similarly, the two observers can parallel transport orthonormal tetrads along their respective worldlines. Comparing these tetrads by means of parallel transport at early and late times, they are related to each other by a linear transformation that is influenced by the burst of gravitational waves. The four-dimensional matrix representing this linear transformation is in general a Lorentz transformation; thus, we will call the effect a *persistent Lorentz transformation observable*. Subtracting from this matrix the identity, one finds that the difference between the initial and final tetrad is given by

$$\Delta\Omega^{\mu}{}_{\nu}(\tau_1,\tau_0) = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 R^{\mu}{}_{\nu\alpha\beta}(\tau_2)\xi^{\alpha}(\tau_0)\dot{\gamma}^{\beta} + O(\boldsymbol{\xi}^2,\boldsymbol{R}^2). \tag{4.5}$$

Note this effect includes the relative velocity observable in the form of a boost (when contracted into $\dot{\gamma}^{\nu}$), as well as a *relative rotation observable*. The relative rotation observable can also be determined by integrating the equation of differential frame dragging (or differential Fokker precession) [62, 123].

Finally, one might wonder if closely-separated geodesic observers measure equal amounts of proper time elapsing along their worldlines during a burst of gravitational waves. In our definition of the displacement memory above, we parametrized the geodesic γ by its proper time, and so we do the same with $\bar{\gamma}$. Suppose that the two observers synchronize their clocks such that the initial connecting geodesic, which defines the initial separation, passes through $\gamma(\tau_0)$ and $\bar{\gamma}(\tau_0)$. The second of these connecting geodesics (defining the final separation) will, in general, pass through the points $\gamma(\tau_1)$ and $\bar{\gamma}[\tau_1 + \Delta\tau(\tau_0, \tau_1)]$, not $\bar{\gamma}(\tau_1)$. The proper time difference is this quantity $\Delta\tau(\tau_0, \tau_1)$, a function both of the initial synchronization time τ_0 and the final time τ_1. Strominger and Zhiboedov considered this observable in [157], which we will call the *persistent relative proper time observable*. Performing a calculation similar to that which yields equation (4.2), we find that

$$\Delta\tau(\tau_1, \tau_0) = \frac{1}{2}\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 R_{\alpha\beta\gamma\delta}(\tau_2)\xi^\alpha\dot{\gamma}^\beta\xi^\gamma\dot{\gamma}^\delta + O(\boldsymbol{\xi}^3, \boldsymbol{R}^2). \tag{4.6}$$

The observables that are discussed in this section are typically discussed in the context of null infinity, where there is also an unambiguous notion of what one means by "nonradiative". In this context, the persistent observables can be expanded in a series in $1/r$, where r is the Bondi-Sachs radial coordinate [37, 143]. Near null infinity, the displacement memory scales as $1/r$ (see, for example, [53, 32]). We are not aware of calculations of the scaling of the relative velocity, rotation, and proper time observables with r near null infinity, but otherwise in fully nonlinear regime. Specializing to linearized gravity, however, one can use the results of [32] to argue that these three persistent observables scale as $1/r^2$ (that is, the term in the expansion at $1/r$ vanishes).[1]

Note that the persistent relative proper time, relative velocity, and relative rotation observables have all been called "memories" previously in the literature. Since they are not associated with symmetries at boundaries of spacetime, we will be referring to them simply as persistent observables.

4.1.2 Curve deviation

We now define our first new persistent observable—which we call *curve deviation*—as a generalization of geodesic deviation. Since geodesic deviation forms the basis of the displacement memory

[1] For spacetimes that are not asymptotically flat in the usual sense, see [58] for an example where the relative velocity does not have this scaling.

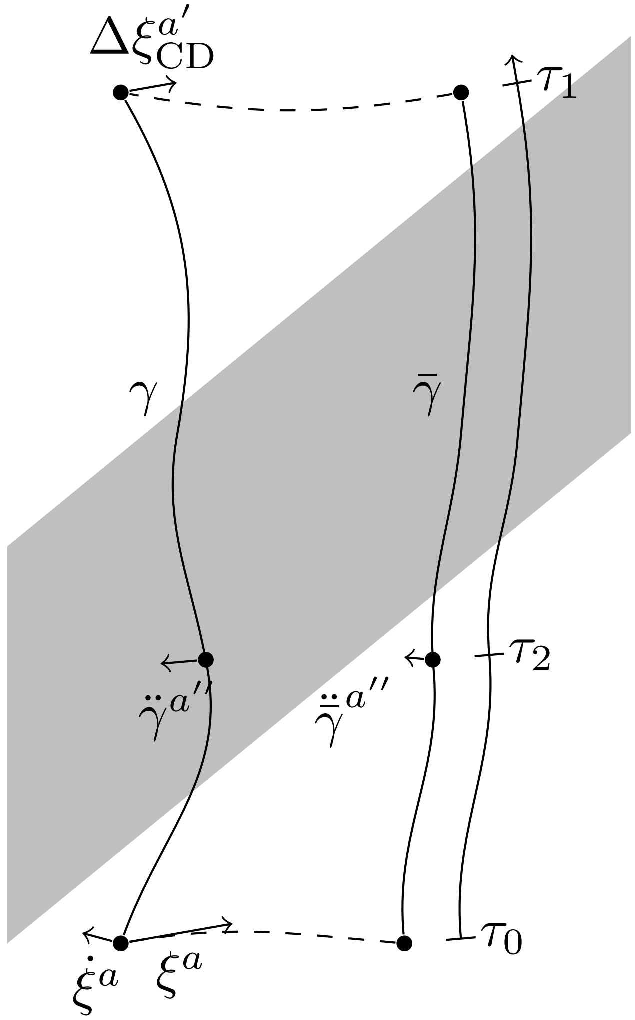

Figure 4.1: Two curves that have some initial separation ξ^a and relative velocity $\dot{\xi}^a$, as well as accelerations $\ddot{\gamma}^{a''}$ and $\ddot{\bar{\gamma}}^{\bar{a}''}$. The curve deviation observable $\Delta\xi^{a'}_{\rm CD}$ is given by the difference between the measured final separation and the final separation that would be predicted if the gray region (containing gravitational waves) were also flat. The dashed lines in this figure, as with those throughout this chapter, refer to the unique geodesics connecting the two endpoints

(both leading and subleading), relative velocity, and relative proper time observables, this observable can be seen as a generalization of all of these previous observables. In particular, it contains the information that is present in each of these individual observables.

We define this observable as follows. Consider two timelike curves γ and $\bar{\gamma}$ that pass through the region of gravitational waves, as depicted in figure 4.1. Let τ_0 denote a value of the proper time along γ and $\bar{\gamma}$ before the gravitational waves have passed, and let τ_1 be a value of proper time for both γ and $\bar{\gamma}$ after the passage of the waves. At any proper time τ, we define the separation vector as tangent to the unique geodesic that connects $\gamma(\tau)$ and $\bar{\gamma}(\tau)$. The given variables in this problem are the initial separation ξ^a and initial relative velocity $\dot{\xi}^a$ at $x \equiv \gamma(\tau_0)$, as well as the accelerations of γ and $\bar{\gamma}$ at all values of proper time.

The curve deviation observable $\Delta\xi^{a'}_{\rm CD}$ at $x' \equiv \gamma(\tau_1)$ is the difference between the actual, measured separation, and the separation predicted from the observers' initial separation, relative velocity, and their measured accelerations, assuming that the region is flat. Explicitly, this observable is defined by

$$\Delta\xi^{a'}_{\rm CD} \equiv \xi^{a'} - {}_\gamma g^{a'}{}_a \left[\xi^a + (\tau_1 - \tau_0)\dot{\xi}^a\right] - \int_{\tau_0}^{\tau_1} {\rm d}\tau_2 \int_{\tau_0}^{\tau_2} {\rm d}\tau_3 \, {}_\gamma g^{a'}{}_{a'''} \left(g^{a'''}{}_{\bar{a}'''}\ddot{\bar{\gamma}}^{\bar{a}'''} - \ddot{\gamma}^{a'''}\right), \qquad (4.7)$$

where the points x''' and $\bar{x}'''$, where the accelerations $\ddot{\gamma}^{a'''}$ and $\ddot{\bar{\gamma}}^{a'''}$ are measured, are defined by $x''' \equiv \gamma(\tau_3)$ and $\bar{x}''' \equiv \bar{\gamma}(\tau_3)$. The quantities ${}_\gamma g^{a'}{}_a$ and $g^{\bar{a}}{}_a$ are so-called *parallel propagators*, and will be defined in section 4.2.1.2. These quantities are *bitensors*, a generalization of tensor fields that depend on two points, which are properly introduced in section 4.2.1. The curve deviation has the property that it will vanish in flat spacetimes, even for arbitrarily accelerating curves, and so is a persistent observable.

Our observable has, in general, a nonlinear dependence on the initial separation and relative velocity, as well as the accelerations. For simplicity, we specialize to small separations, relative velocities, and accelerations, and expand the curve deviation as

$$\begin{aligned}\Delta\xi^{a'}_{\rm CD} \equiv &\left[\Delta K^{a'}{}_b + L^{a'}{}_{bc}\xi^c + N^{a'}{}_{bc}\dot{\xi}^c + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2\right]\xi^b + \left[(\tau_1-\tau_0)\Delta H^{a'}{}_b + M^{a'}{}_{bc}\dot{\xi}^c + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2\right]\dot{\xi}^b \\ &+ \int_{\tau_0}^{\tau_1} {\rm d}\tau_2 (\tau_1-\tau_2)\Delta H^{a'}{}_{b''}\left\{g^{b''}{}_{\bar{b}''}\left[1 + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2\right]\ddot{\bar{\gamma}}^{\bar{b}''} - \left[1+O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2\right]\ddot{\gamma}^{b''}\right\} + O(\ddot{\gamma},\ddot{\bar{\gamma}})^2,\end{aligned} \qquad (4.8)$$

for some bitensors $\Delta K^{a'}{}_a$, $L^{a'}{}_{bc}$, $N^{a'}{}_{bc}$, $\Delta H^{a'}{}_a$, and $M^{a'}{}_{bc}$, and where we define $x'' \equiv \gamma(\tau_2)$. The bitensors $\Delta K^{a'}{}_a$, $\Delta H^{a'}{}_a$, $L^{a'}{}_{bc}$, $M^{a'}{}_{bc}$, and $N^{a'}{}_{bc}$ vanish in flat spacetime and are determined by the curve γ (and therefore depend, implicitly, on the acceleration $\ddot{\gamma}^a$). Here, as mentioned above, we are using $O(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})^2$ as shorthand for $O(\boldsymbol{\xi}^2, \boldsymbol{\xi} \cdot \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}^2)$. The fact that this observable has an expansion in the form of equation (4.8) is non-trivial; a derivation of this result is given in section 4.2.3.1, and explicit expressions for all of the bitensors that this expression defines are given in equation (4.143).

We now make a note about comparing the definitions in this section with those that occur in section 4.1.1. In order to define the separation vector, one needs to choose which points on the two worldlines correspond to one another: such a choice is known as a *correspondence* [173]. In section 4.1.1, the separation vector was defined using the unique geodesic which connected the two worldlines of the observers and was orthogonal to one of the worldlines; this choice is the *normal correspondence*. On the other hand, in this section, the separation vector is defined using the unique geodesic that connects the two worldlines at equal proper times; this choice is the *isochronous correspondence*. This latter correspondence is only unique up to initial synchronization. As we will review in section 4.2.2.1, computations in the isochronous correspondence are far easier, and so from this point forward we will be using this correspondence exclusively. For further discussion, see [173].

At first glance, it does not seem possible for an observable defined using the isochronous correspondence to contain any information about the relative proper time observable of section 4.1.1, since the proper times of the two observers are always synchronized. This, however, is not correct: while the proper time difference vanishes in the isochronous correspondence, the final separation is no longer orthogonal to γ, as it was in the case of the normal correspondence. In fact, one can show that (in the case of vanishing acceleration after the burst)

$$\Delta\tau(\tau_1, \tau_0) = \frac{\dot{\gamma}_{a'}\xi^{a'}}{1 - \dot{\gamma}_{b'}\dot{\xi}^{b'}} = \frac{\dot{\gamma}_{a'}\Delta\xi^{a'}_{\rm CD}}{1 - \dot{\gamma}_{b'}\Delta\dot{\xi}^{b'}_{\rm CD}}, \tag{4.9}$$

where $\Delta\tau(\tau_1, \tau_0)$ is the proper time delay in section 4.1.1, and $\Delta\dot{\xi}^{a'}_{\rm CD}$ is the time derivative of the curve deviation observable. As such, $\dot{\gamma}_{a'}\Delta\xi^{a'}_{\rm CD}$ [which, as one can show, is $O(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})^2$] is, to leading order, the proper time observable.

As with the previous examples in the literature, we also present the following results that are valid to linear order in the Riemann tensor. To this order, the bitensors in equation (4.8) are given by

$$\Delta K^{\alpha}{}_{\beta}(\tau_1, \tau_0) = -\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_0}^{\tau_2} \mathrm{d}\tau_3 R^{\alpha}{}_{\gamma\beta\delta}(\tau_3)\dot{\gamma}^{\gamma}(\tau_3)\dot{\gamma}^{\delta}(\tau_3) + O(\boldsymbol{R}^2), \tag{4.10a}$$

$$\Delta H^{\alpha}{}_{\beta}(\tau_1, \tau_0) = -\frac{1}{\tau_1 - \tau_0}\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_0}^{\tau_2} \mathrm{d}\tau_3 \int_{\tau_3}^{\tau_2} \mathrm{d}\tau_4 R^{\alpha}{}_{\gamma\beta\delta}(\tau_4)\dot{\gamma}^{\gamma}(\tau_4)\dot{\gamma}^{\delta}(\tau_4) + O(\boldsymbol{R}^2), \tag{4.10b}$$

$$L^{\alpha}{}_{\beta\gamma}(\tau_1, \tau_0) = -\frac{1}{2}\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_0}^{\tau_2} \mathrm{d}\tau_3 S^{\alpha}{}_{\beta\gamma\delta\epsilon}(\tau_3)\dot{\gamma}^{\delta}(\tau_3)\dot{\gamma}^{\epsilon}(\tau_3) + O(\boldsymbol{R}^2), \tag{4.10c}$$

$$N^{\alpha}{}_{\beta\gamma}(\tau_1, \tau_0) = -\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_0}^{\tau_2} \mathrm{d}\tau_3 \left[\int_{\tau_3}^{\tau_2} \mathrm{d}\tau_4 S^{\alpha}{}_{\beta\gamma\delta\epsilon}(\tau_4)\dot{\gamma}^{\epsilon}(\tau_4)\dot{\gamma}^{\delta}(\tau_4) + 2R^{\alpha}{}_{\gamma\beta\delta}(\tau_3)\dot{\gamma}^{\delta}(\tau_3)\right] + O(\boldsymbol{R}^2), \tag{4.10d}$$

$$\begin{aligned} M^{\alpha}{}_{\beta\gamma}(\tau_1, \tau_0) = &\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_0}^{\tau_2} \mathrm{d}\tau_3 \int_{\tau_3}^{\tau_2} \mathrm{d}\tau_4 \left[\frac{1}{2}\int_{\tau_3}^{\tau_4} \mathrm{d}\tau_5 S^{\alpha}{}_{\beta\gamma\delta\epsilon}(\tau_5)\dot{\gamma}^{\delta}(\tau_5)\dot{\gamma}^{\epsilon}(\tau_5) - 2R^{\alpha}{}_{(\gamma\beta)\delta}(\tau_4)\dot{\gamma}^{\delta}(\tau_4)\right] \\ &- \frac{1}{2}\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_2}^{\tau_1} \mathrm{d}\tau_3 \int_{\tau_0}^{\tau_3} \mathrm{d}\tau_4 \int_{\tau_4}^{\tau_3} \mathrm{d}\tau_5 S^{\alpha}{}_{\beta\gamma\delta\epsilon}(\tau_5)\}\dot{\gamma}^{\delta}(\tau_5)\dot{\gamma}^{\epsilon}(\tau_5) + O(\boldsymbol{R}^2), \end{aligned} \tag{4.10e}$$

where

$$S^{a}{}_{bcde} \equiv \nabla_{(e}R^{a}{}_{b)cd} + \nabla_{(e}R^{a}{}_{c)bd}. \tag{4.11}$$

Note that the first of these expressions is very similar to equation (4.2), and the second is very similar to equation (4.3). In these expressions, the parallel-transported components of the four-velocity are not constant functions of proper time, as γ is not necessarily geodesic.

4.1.3 Holonomy observables and angular momentum transport

The next set of observables that we introduce are a type of holonomy observable; that is, observables that arise due to solving an ordinary differential equation around a closed curve. An observable of this type was already introduced above, in the form of the relative Lorentz transformation observable in section 4.1.1. There, the differential equation that was being solved was that of parallel transport. In this section, we first review a more general type of holonomy introduced in [68] that is known as the *generalized holonomy*, and then show that it is related to another type of holonomy that is related to the transport of angular momentum in curved spacetimes.

4.1.3.1 Generalized holonomy

We start with the generalized holonomy of [68], which is a covariant observable that encodes the four persistent gravitational wave observables of section 4.1.1 (displacement, velocity, proper time, and rotation) in a single quantity. Consider a solution χ^a of an *affine transport* law along a curve with tangent vector k^a:

$$k^a \nabla_a \chi^b = -k^b. \tag{4.12}$$

If one solves equation (4.12) with a given initial χ^a at some point x, then the final $\chi^{a'}$ at some point x' along the curve can be written as follows:

$$\chi^{a'} = g^{a'}{}_a \chi^a + \Delta\chi^{a'}(x). \tag{4.13}$$

Note the dependence on the initial position in $\Delta\chi^{a'}(x)$. The homogeneous solution $g^{a'}{}_a \chi^a$ corresponds to parallel transport of the given initial vector χ^a, in terms of the parallel propagator $g^{a'}{}_a$ (see section 4.2.1.2 for more details). The inhomogeneous solution $\Delta\chi^{a'}(x)$ generalizes the notion of a separation vector between two points in flat spacetime. In a curved spacetime, $\Delta\chi^{a'}(x)$ and $g^{a'}{}_a$ depend on the curve connecting the points x and x'.

Consider now a closed curve composed of two initially comoving timelike geodesics γ and $\bar{\gamma}$, along with two spatial geodesics connecting γ and $\bar{\gamma}$ at the initial and final points of γ and $\bar{\gamma}$. Solve equation (4.12) around this curve by starting at the initial point of γ, evolving forwards along γ, then along the geodesic connecting the final points, then backwards along $\bar{\gamma}$, and then finally along the geodesic between the initial points (this is the same orientation as given in figure 4.2, which is introduced in section 4.1.3.2). The solution (4.13) defines a mapping

$$\chi^a \to \Lambda^a{}_b(\gamma, \bar{\gamma}; \tau_1)\chi^b + \Delta\chi^a(\gamma, \bar{\gamma}; \tau_1). \tag{4.14}$$

The two observables on the right-hand side of this equation form the generalized holonomy of [68]. Here, $\Lambda^a{}_b(\gamma, \bar{\gamma}; \tau_1)$ is the usual holonomy around this curve with respect to parallel transport.

It was shown in [68] that this generalized holonomy encodes the displacement memory, in addition to the persistent relative velocity, rotation, and proper time observables. Using the same normal correspondence as was used to define those observables in section 4.1.1, one finds (on a parallel-transported basis) [2]

$$\Lambda^{\mu}{}_{\nu}(\gamma, \bar{\gamma}; \tau_1, \tau_0) = \delta^{\mu}{}_{\nu} + \Delta\Omega^{\mu}{}_{\nu}(\tau_1, \tau_0), \tag{4.15a}$$

$$\begin{aligned}\Delta\chi^{\mu}(\gamma, \bar{\gamma}; \tau_1, \tau_0) = &-\Delta\Omega^{\mu}{}_{\nu}(\tau_1, \tau_0)\left[\xi^{\nu}(\tau_0) + (\tau_1 - \tau_0)\dot{\gamma}^{\nu}\right] + \Delta\tau(\tau_1, \tau_0)\dot{\gamma}^{\mu} \\ &- \Lambda^{\mu}{}_{\nu}(\gamma, \bar{\gamma}; \tau_1, \tau_0)\Delta\xi^{\nu}(\tau_1, \tau_0).\end{aligned} \tag{4.15b}$$

The latter of these two equations, in the isochronous correspondence and assuming that γ and $\bar{\gamma}$ are geodesic, takes the form

$$\Delta\chi^{a}(\gamma, \bar{\gamma}; \tau_1) = \xi^{a} - \Lambda^{a}{}_{b}(\gamma, \bar{\gamma}; \tau_1)\,{}_{\gamma}g^{b}{}_{b'}\left[\xi^{b'} - (\tau_1 - \tau_0)\dot{\xi}^{b'}\right]. \tag{4.16}$$

This equation is a fully *non-perturbative* relationship between the inhomogeneous and homogeneous pieces of the generalized holonomy.

4.1.3.2 Angular momentum transport

Our next observable is motivated by the rough argument that we gave at the end of section 1.3.3.2 that relates angular momentum in flat spacetime to the displacement memory observable. Another motivation comes from the fact that the affine transport law introduced in section 4.1.3.1 also defines a means of relating linear and angular momenta at different points [68]. Here, we mean either the linear and angular momentum of some extended body, or the linear and angular momentum of the spacetime itself. There are a variety of prescriptions by which an observer could define linear and angular momentum, but what is important here is how such an observer would sensibly transport these quantities from point to point.

As a motivating example, consider a freely falling body in flat spacetime. Stated in terms of the solutions to the affine transport law in equation (4.12), an observer at some point x would measure the total linear and angular momentum of the body (about their location) to be

$$P^{a} \equiv g^{a}{}_{a'}P^{a'}, \tag{4.17a}$$

$$J^{ab} \equiv g^{a}{}_{a'}g^{b}{}_{b'}S^{a'b'} + 2\Delta\chi^{[a}(x')P^{b]}, \tag{4.17b}$$

[2] Note that this result is not the same equation as that given in [68] [their equation (3.20)], because we are using a different initial point and direction for traversing the loop. The two results are consistent.

where x' is a point on the body's center of mass worldline that intersects a geodesic orthogonal to the observer's worldline at x, and $S^{a'b'}$ and $P^{a'}$ are the intrinsic angular momentum and linear momentum of the body at x'. The first term in equation (4.17b) is the intrinsic angular momentum, and the second term is the orbital angular momentum, as it depends on the separation of the observer relative to the body. Parallel propagators appear in this expression to reflect the fact that $P^{a'}$ and $S^{a'b'}$, as tensor fields, are only defined on the center of mass worldline of the body—in flat spacetime, these propagators are trivial, but it is useful to have them explicitly in these expressions.

A crucial feature of this example is that (when the intrinsic linear and angular momentum of the body are conserved) the constructed P^a and J^{ab} obey the following coupled differential equations along an arbitrary curve with tangent vector k^a:

$$k^a \nabla_a P^b = 0, \tag{4.18a}$$

$$k^a \nabla_a J^{bc} = 2P^{[b} k^{c]}. \tag{4.18b}$$

These differential equations capture the "origin dependence" of angular momentum in special relativity: shifting the origin about which we are measuring the angular momentum along some curve with tangent vector k^a, equation (4.18) tells us how the linear and angular momentum change. This origin dependence is a crucial feature of angular momentum, and so one could consider equation (4.18) as the *definition* of how linear and angular momentum should be transported in curved spacetimes. Unlike in flat spacetime, this transport depends on the curve that is used between the two points.

This is just one method of angular momentum transport, and is the example with minimal coupling to gravity. Reference [69] introduced a more general version of this transport law, which we later extended in [67]. Instead of equation (4.18), we consider the following differential equation:

$$k^b \nabla_b P^a = -\overset{\varkappa}{K}{}^a{}_{bcd} k^b J^{cd}, \tag{4.19a}$$

$$k^c \nabla_c J^{ab} = 2P^{[a} k^{b]}. \tag{4.19b}$$

Here $\varkappa = (\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4)$ is a collection of constant *transport parameters*, and the tensor $\overset{\varkappa}{K}{}^{ab}{}_{cd}$ is defined by

$$\overset{\varkappa}{K}{}^{ab}{}_{cd} = \varkappa_1 R^{ab}{}_{cd} + \varkappa_2 \delta^a{}_{[c} R^b{}_{d]} + \varkappa_3 \delta^b{}_{[c} R^a{}_{d]} + \varkappa_4 R \delta^a{}_{[c} \delta^b{}_{d]}. \tag{4.20}$$

Before [67], two special cases of this transport law had been considered: $\varkappa = (0, 0, 0, 0)$, which was studied in [68], and $\varkappa = (\kappa, 0, 0, 0)$, which was studied in [69].

Before describing the holonomy observable for angular momentum transport, we first make our notation more compact. The idea is that the transport law described in equation (4.19) can be written as parallel transport of a connection defined on a ten dimensional vector bundle, which we call the *linear and angular momentum bundle*. This vector bundle is constructed by taking, at each point, the direct sum of two vector spaces, one from the tangent bundle, and the other of bivectors, or antisymmetric rank $(2, 0)$ tensors:

$$\mathcal{A} = T\mathcal{M} \oplus \Lambda^2 T\mathcal{M}. \tag{4.21}$$

This obscure mathematical definition just means that the objects that we are considering are ten-dimensional vectors of the form

$$X^A = \begin{pmatrix} P^a \\ J^{ab} \end{pmatrix}, \tag{4.22}$$

for a vector P^a and an antisymmetric tensor J^{ab}. For any matrix on this vector bundle, we denote the various components as follows:

$$A^A{}_C \equiv \begin{pmatrix} \underset{PP}{A}{}^a{}_c & \underset{PJ}{A}{}^a{}_{cd} \\ \underset{JP}{A}{}^{ab}{}_c & \underset{JJ}{A}{}^{ab}{}_{cd} \end{pmatrix}. \tag{4.23}$$

We discuss a useful way to perform algebraic decompositions of these matrices in appendix 4.B.

The above notation clearly makes any equations involving both linear and angular momentum far more compact. Moreover, there is a natural notion of a connection on this bundle that is associated with the transport law in equation (4.19). This equation can be rewritten as

$$0 = k^a \overset{\varkappa}{\nabla}_a X^B \equiv k^a \nabla_a X^B + k^a \overset{\varkappa}{C}{}^B{}_{Ca} X^C, \tag{4.24}$$

where ∇_a denotes the natural extension of the metric-compatible connection to this bundle, and

$$\overset{\varkappa}{C}{}^A{}_{Ce} = \begin{pmatrix} 0 & \overset{\varkappa}{K}{}^a{}_{ecd} \\ 2\delta^{[a}{}_e \delta^{b]}{}_c & 0 \end{pmatrix}. \tag{4.25}$$

This one-form-valued matrix on the linear and angular momentum bundle forms the connection coefficients between $\overset{\varkappa}{\nabla}_a$ and ∇_a. Equation (4.24) makes it clear that solving equation (4.19) is the

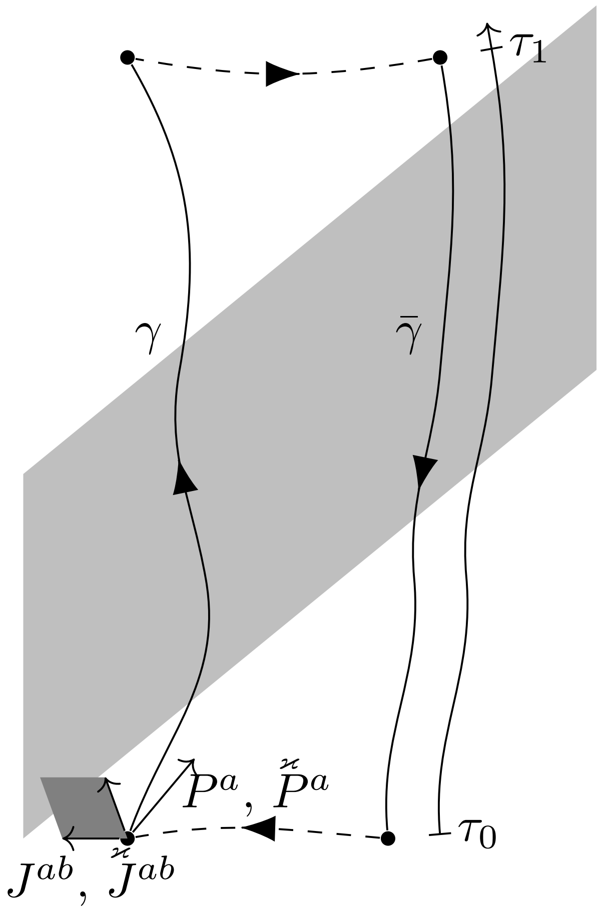

Figure 4.2: A loop about which observers can compute a holonomy that measures the effect of a burst of gravitational waves. The quantities P^a and J^{ab} are transported around this loop using equation (4.19) (with some set of parameters $\varkappa$) in the directions shown, thereby yielding the observables $\overset{\varkappa}{P}^a$ and $\overset{\varkappa}{J}^{ab}$.

same as parallel-transporting X^A along the curve with tangent vector k^a. In particular, this means that one can solve this equation around a loop using tools for computing holonomies for arbitrary connections on vector bundles.

At this point, we introduce the *angular momentum holonomy observable* $\overset{\varkappa}{\Lambda}{}^A{}_B(\gamma, \bar{\gamma}; \tau_1)$ by solving equation (4.19) around the closed, narrow loop in figure 4.2 defined for observers following γ and $\bar{\gamma}$. This gives us a curve-dependent observable (in the form of a matrix at a given point) describing how the final linear and angular momentum, which we denote by $\overset{\varkappa}{P}^a$ and $\overset{\varkappa}{J}^{ab}$, depend on the initial linear and angular momentum, which we denote by P^a and J^{ab}:

$$\overset{\varkappa}{X}{}^A = \overset{\varkappa}{\Lambda}{}^A{}_B(\gamma, \bar{\gamma}; \tau_1) X^A, \tag{4.26}$$

where $\overset{\varkappa}{X}^A$ is defined analogously to equation (4.22). This observable depends on the *curve* used in its definition; we note that in [69], this holonomy was instead computed for infinitesimal square loops.

The observable in equation (4.26) is not truly a persistent observable in the context described in this chapter, as it is nonzero even in the absence of gravitational waves. However, it is *trivial*, acting simply as the identity matrix on the linear and angular momentum bundle. As such, we find it convenient to define

$$\overset{\varkappa}{\Omega}{}^A{}_B(\gamma, \bar{\gamma}; \tau_1) \equiv \overset{\varkappa}{\Lambda}{}^A{}_B(\gamma, \bar{\gamma}; \tau_1) - \delta^A{}_B, \tag{4.27}$$

as the persistent observable associated with the holonomy. As we will show in section 4.2.2.2, the holonomy depends on the separation vector ξ^a throughout the curved region, so it depends on the initial separation, relative velocity, and the accelerations of the curves. The particular dependence is of the following form, to leading order:

$$\overset{\varkappa}{\Omega}{}^A{}_B(\gamma, \bar{\gamma}; \tau_1) = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \left[\overset{\varkappa}{\Omega}{}^A{}_{Bc''d''}(\gamma)\dot{\gamma}^{[c''}\xi^{d'']} + O(\boldsymbol{\xi}'', \dot{\boldsymbol{\xi}}'')^2\right], \tag{4.28}$$

for a new bitensor $\overset{\varkappa}{\Omega}{}^A{}_{Bc''d''}(\gamma)$ which vanishes in flat regions. As such, $\overset{\varkappa}{\Omega}{}^A{}_B(\gamma, \bar{\gamma}; \tau_1)$ is clearly an integrated measure of curvature.

For a general set of parameters $\varkappa$, our results are given in equation (4.152). For convenience, we list our results in the weak curvature limit below for general $\varkappa$:

$$\begin{aligned}\overset{\varkappa}{\Omega}{}^\Gamma{}_\Delta(\gamma, \bar{\gamma}; \tau_1) = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \Bigg\{ & \overset{\varkappa}{\Phi}{}^\Gamma{}_{\Delta\kappa}(\gamma; \tau_1, \tau_2)\xi^\kappa + \int_{\tau_2}^{\tau_1} \mathrm{d}\tau_3 \overset{\varkappa}{\Phi}{}^\Gamma{}_{\Delta\kappa}(\gamma; \tau_1, \tau_3)\dot{\xi}^\kappa \\ & + [\ddot{\bar{\gamma}}^\kappa(\tau_2) - \ddot{\gamma}^\kappa(\tau_2)] \int_{\tau_2}^{\tau_1} \mathrm{d}\tau_3 \int_{\tau_3}^{\tau_1} \mathrm{d}\tau_4 \overset{\varkappa}{\Phi}{}^\Gamma{}_{\Delta\kappa}(\gamma; \tau_1, \tau_4)\Bigg\} \\ & + O(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})^2 + O(\boldsymbol{R}, \boldsymbol{\nabla R})^2,\end{aligned} \tag{4.29}$$

where

$$\underset{PP}{\Phi}{}^\alpha{}_{\mu\kappa}(\tau_1, \tau_2) = \left\{R^\alpha{}_{\mu\kappa\lambda}(\tau_2) + 4\overset{\varkappa}{K}{}^\alpha{}_{(\mu|\lambda|\kappa)}(\tau_2) + 2\int_{\tau_2}^{\tau_1} \mathrm{d}\tau_3 [\nabla_\kappa \overset{\varkappa}{K}{}^\alpha{}_{\sigma\mu\lambda}](\tau_3)\dot{\gamma}^\sigma(\tau_3)\right\}\dot{\gamma}^\lambda(\tau_2), \tag{4.30a}$$

$$\underset{PJ}{\Phi}{}^{\alpha}{}_{\mu\nu\kappa}(\tau_1,\tau_2) = 2[\nabla_{[\kappa}\breve{\breve{K}}{}^{\alpha}{}_{\lambda]\mu\nu}](\tau_2)\dot{\gamma}^{\lambda}(\tau_2), \tag{4.30b}$$

$$\begin{aligned}\underset{JP}{\Phi}{}^{\alpha\beta}{}_{\mu\kappa}(\tau_1,\tau_2) = 8\int_{\tau_2}^{\tau_1}\mathrm{d}\tau_3\Bigg(&\delta^{[\alpha}{}_{\rho}\dot{\gamma}^{\beta]}(\tau_2)\Bigg\{[\nabla_{[\kappa}\breve{\breve{K}}{}^{\rho}{}_{\lambda]\mu\zeta}](\tau_3)\int_{\tau_0}^{\tau_2}\mathrm{d}\tau_4\dot{\gamma}^{\zeta}(\tau_4) - \breve{\breve{K}}{}^{\rho}{}_{(\mu|\lambda|\kappa)}(\tau_3) \\ &\qquad - \frac{1}{2}\int_{\tau_3}^{\tau_1}\mathrm{d}\tau_4[\nabla_{\kappa}\breve{\breve{K}}{}^{\rho}{}_{\sigma\mu\lambda}](\tau_4)\dot{\gamma}^{\sigma}(\tau_4)\Bigg\} \\ &+ \frac{1}{4}\dot{\gamma}^{\sigma}(\tau_2)\Big[\delta^{[\alpha}{}_{\mu}R^{\beta]}{}_{\sigma\kappa\lambda}(\tau_3) + 4\delta^{[\alpha}{}_{[\kappa}\breve{\breve{K}}{}^{\beta]}{}_{\lambda]\mu\sigma}(\tau_3)\Big]\Bigg)\dot{\gamma}^{\lambda}(\tau_3),\end{aligned} \tag{4.30c}$$

$$\begin{aligned}\underset{JJ}{\Phi}{}^{\alpha\beta}{}_{\mu\nu\kappa}(\tau_1,\tau_2) = {}& 2\dot{\gamma}^{[\alpha}(\tau_2)\int_{\tau_2}^{\tau_1}\mathrm{d}\tau_3[\nabla_{\kappa}\breve{\breve{K}}{}^{\beta]}{}_{\lambda\mu\nu}](\tau_3)\dot{\gamma}^{\lambda}(\tau_3) \\ &+ 2\Big[\delta^{[\alpha}{}_{[\mu}R^{\beta]}{}_{\nu]\kappa\lambda}(\tau_2) + \delta^{[\alpha}{}_{\kappa}\breve{\breve{K}}{}^{\beta]}{}_{\lambda\mu\nu}(\tau_2)\Big]\dot{\gamma}^{\lambda}(\tau_2).\end{aligned} \tag{4.30d}$$

These expressions are quite complicated for the case of general $\varkappa$, even in this weak curvature limit. In appendix 4.B, we will consider a way of decomposing the holonomy into different parts, which may aid in understanding the meaning of the large number of components in the holonomy.

4.1.3.3 Specific values of the transport parameters

We now consider particular values of $\varkappa$. As mentioned above, the holonomy for $\varkappa = (0,0,0,0)$, which we will refer to as *affine transport*, can be written in terms of the generalized holonomy. Denoting by $\overset{0}{\nabla}_a$ the connection on the linear and angular momentum bundle that is used for affine transport, and using a "0" diacritic for all quantities related to this connection, we have that

$$\overset{0}{\Lambda}{}^{A}{}_{C}(\gamma,\bar{\gamma};\tau_1) = \begin{pmatrix} \Lambda^{a}{}_{c}(\gamma,\bar{\gamma};\tau_1) & 0 \\ 2\Delta\chi^{[a}(\gamma,\bar{\gamma};\tau_1)\Lambda^{b]}{}_{c}(\gamma,\bar{\gamma};\tau_1) & \Lambda^{[a}{}_{c}(\gamma,\bar{\gamma};\tau_1)\Lambda^{b]}{}_{d}(\gamma,\bar{\gamma};\tau_1)\end{pmatrix}. \tag{4.31}$$

Thus, the value of this holonomy has already been effectively computed in [68], and (as noted in section 4.1.3.1) the components of this holonomy describe various observables of section 4.1.1. This form of the affine transport holonomy can be thought of as a Poincaré transformation, as $\Delta\chi^a(\gamma,\bar{\gamma};\tau_1)$ is a vector and $\Lambda^a{}_b(\gamma,\bar{\gamma};\tau_1)$ a Lorentz transformation [68]. While equation (4.31) is convenient for non-perturbative calculations (such as those in section 4.3.3.2), this method of computing our holonomy observable does not extend readily to other values of $\varkappa$. As such, in section 4.2.3.2, we will calculate this holonomy in a framework which works for all values of $\varkappa$, obtaining our final answer in equation (4.156).

We also consider the case $\varkappa = (1/2, 0, 0, 0)$, which we will refer to as *dual Killing transport*; our final answer for the holonomy for this particular value of $\varkappa$ is given in (4.168). The connection for this transport law will be denoted by $\overset{1/2}{\nabla}_a$, and quantities relying on this particular value of $\varkappa$ will have a "1/2" diacritic placed above them. The holonomy of dual Killing transport describes how the space of symmetries changes because of the burst of gravitational waves. This is due to the relationship between the transport law in equation (4.19) with $\varkappa = (1/2, 0, 0, 0)$ and the Killing transport equations which determine how Killing vector fields can be determined from initial data at a point. This can be seen as follows: for any Killing vector field ξ^a, define $\omega_{ab} \equiv \nabla_{[a}\xi_{b]}$. The Killing transport equations can then be written as

$$\nabla_a \xi_b = \omega_{ab}, \tag{4.32a}$$

$$\nabla_a \omega_{bc} = R^d{}_{abc}\xi_d. \tag{4.32b}$$

Defining a covector Y_A on the linear and angular momentum bundle by

$$Y_A \equiv \begin{pmatrix} \xi_a & \frac{1}{2}\omega_{ab} \end{pmatrix}, \tag{4.33}$$

one can easily see that the Killing transport equations are equivalent to

$$\overset{1/2}{\nabla}_a Y_B = 0. \tag{4.34}$$

As such, dual Killing transport is dual to Killing transport. A consequence of this is that, for P^a and J^{ab} transported along a curve by dual Killing transport,

$$Q \equiv X^A Y_A = P^a \xi_a + \frac{1}{2} J^{ab}\omega_{ab} \tag{4.35}$$

is a constant (see, for example, [87]).

This association between linear and angular momentum and Killing vector fields allows one to think of the holonomy of dual Killing transport in a slightly different way. Consider a Killing vector field ξ^a in the initial flat region I. The value of this vector field and its derivative at $\gamma(\tau_0)$ can be used as initial data for Killing transport along γ, through region II, and into the final flat region III. The result of this Killing transport at $\gamma(\tau_1)$ can be used to construct another Killing vector field $\xi^{a'}$ in region III. The value of $\xi^{a'}$ and its derivative at $\bar{\gamma}(\tau_1)$ can likewise be used as initial data for

Killing transport along $\bar{\gamma}$, and the result of this Killing transport can be used to construct a Killing vector field $\tilde{\xi}^a$ in region I. One can show that this Killing vector field is related to ξ^a at $\gamma(\tau_0)$ by the same holonomy observable:

$$\tilde{\xi}_a = \overset{1/2}{\underset{PP}{\Lambda}}{}^c{}_a(\gamma, \bar{\gamma}; \tau_1)\xi_c + \frac{1}{2}\overset{1/2}{\underset{JP}{\Lambda}}{}^{cd}{}_a(\gamma, \bar{\gamma}; \tau_1)\nabla_c \xi_d, \tag{4.36a}$$

$$\nabla_a \tilde{\xi}_b = 2\overset{1/2}{\underset{PJ}{\Lambda}}{}^c{}_{ab}(\gamma, \bar{\gamma}; \tau_1)\xi_c + \overset{1/2}{\underset{JJ}{\Lambda}}{}^{cd}{}_{ab}(\gamma, \bar{\gamma}; \tau_1)\nabla_c \xi_d. \tag{4.36b}$$

As such, this holonomy observable can be thought of as measuring how the spaces of Killing vector fields in regions I and III are related. It can also be thought of as the final linear and angular momentum that would arise from using the Mathisson-Papapetrou equations (discussed below in section 4.1.4) to transport these momenta around a closed curve.

Finally, there is a third relevant value of $\varkappa$, namely $\varkappa = (-1/4, 1/2, 0, 0)$. This is the value of $\varkappa$ that is most interesting near null infinity, as discussed in [69], since it turns out that the holonomy with this value of $\varkappa$ is trivial in an asymptotic sense in stationary regions. We will be discussing this in more detail in future work, but an interesting property of this value of $\varkappa$ is that this is the only value of $\varkappa$ that is allowed to have solutions of the form

$$\overset{\varkappa}{\nabla}_a X^B = 0. \tag{4.37}$$

To see this, we start by supposing that equation (4.37) holds, and consider an antisymmetrized second derivative of J^{ab}:

$$2\nabla_{[a}\nabla_{b]}J^{cd} = -2R^{[c}{}_{eab}J^{d]e} = 4\nabla_{[a}P^{[c}\delta^{d]}{}_{b]}. \tag{4.38}$$

Contracting a and c yields

$$-R_{ab}J^{da} + R^d{}_{cab}J^{ac} = \delta^d{}_b\nabla_a P^a + 2\nabla_b P^d \tag{4.39}$$

(after relabeling), and contracting again on b and d yields

$$2R_{ab}J^{ab} = 6\nabla_a P^a. \tag{4.40}$$

Since R_{ab} is symmetric and J^{ab} antisymmetric, this shows that $\nabla_a P^a = 0$. As such, we find that

$$\nabla_b P^a = -\left(\frac{1}{2}R^a{}_{[cd]b} + \frac{1}{2}\delta^a{}_{[c}R_{|b|d]}\right)J^{cd}. \tag{4.41}$$

Now,

$$R^a{}_{[cd]b} = \frac{1}{2}\left(R^a{}_{cdb} + R^a{}_{dbc}\right) = -\frac{1}{2}R^a{}_{bcd}, \tag{4.42}$$

using the algebraic Bianchi identity, and so we find that

$$\begin{aligned} \nabla_b P^a &= -\left(-\frac{1}{4}R^a{}_{bcd} + \frac{1}{2}\delta^a{}_{[c}R_{|b|d]}\right)J^{cd} \\ &\equiv -\overset{\text{-1/4}}{K}{}^a{}_{bcd}J^{cd}. \end{aligned} \tag{4.43}$$

These are, incidentally, the same integrability conditions that occur for closed, conformal Killing-Yano tensors [160], and so this transport law can be referred to as *closed, conformal Killing-Yano (CCKY) transport.*

4.1.4 Spinning particles

The holonomy given in the previous section is a powerful mathematical map for determining how a radiative region has affected how an observer keeps track of some angular momentum that they have measured. It is, however, a very abstract quantity, and rather difficult to measure in general. The curve deviation observable given in section 4.1.2 is a more realistically observable quantity, but it also requires the observers to measure their acceleration at all times. A more ideal observable would be one that would only require measurements before and after a burst of gravitational waves.

An example of such an observable is given by the following procedure: a freely falling observer measures the linear momentum and intrinsic spin of a comoving test particle, in addition to its separation from the observer. The observer and the test particle then travel along their own worldlines, and after the burst of gravitational waves, the separation, linear momentum, and intrinsic spin of the test particle are measured again. The procedure is depicted in figure 4.3, where we denote the worldlines of the observer and spinning test particle by γ and $\bar{\gamma}$, respectively. The differences between the initial and final separations, linear momenta, and spins per unit mass are the natural observables in this procedure.

We note that both the linear momentum and the intrinsic spin are tensors, and therefore, unless the initial and final separations are zero, we must specify a prescription for transporting these tensors away from the worldline of the spinning particle. The convention which we use is that both are parallel-transported along a curve connecting the two worldlines. Since the regions before and

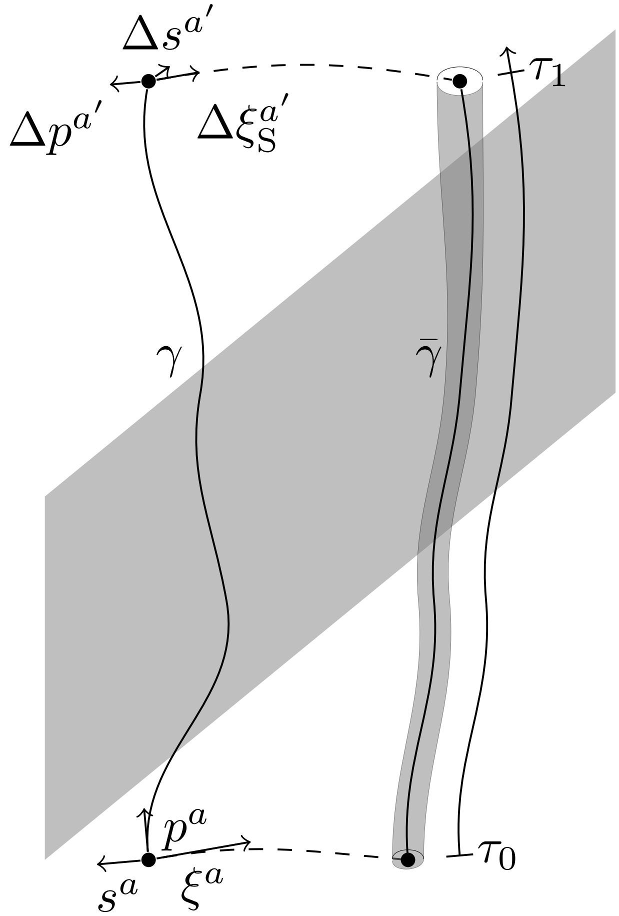

Figure 4.3: An observer (left curve, γ) measuring properties of a spinning test particle (right curve, $\bar{\gamma}$). The test particle has some measured separation ξ^a, linear momentum p^a, and intrinsic spin s^a before a burst of gravitational waves. After the burst, these quantities are all measured again and compared with their values before the burst, yielding $\Delta\xi^{a'}$, $\Delta p^{a'}$, and $\Delta s^{a'}$.

after the burst are flat, this procedure is independent of the particular curves used. Moreover, the measurements of linear momentum, intrinsic spin, and the separation are all tensors at specific points along the observer's worldline, so they must be parallel-transported along the observer's worldline to some common point in order to be compared.

This procedure is qualitatively similar to the holonomy for dual Killing transport, for the following reason: the linear and angular momentum of the spinning test particle evolve along its worldline according to the Mathisson-Papapetrou equations [118, 126]:

$$\dot{\bar{\gamma}}^{\bar{b}}\nabla_{\bar{b}}p^{\bar{a}} = -\frac{1}{2}R^{\bar{a}}{}_{\bar{b}\bar{c}\bar{d}}\dot{\bar{\gamma}}^{\bar{b}}j^{\bar{c}\bar{d}}, \tag{4.44a}$$

$$\dot{\bar{\gamma}}^{\bar{c}}\nabla_{\bar{c}}j^{\bar{a}\bar{b}} = 2p^{[\bar{a}}\dot{\bar{\gamma}}^{\bar{b}]}. \tag{4.44b}$$

Here $p^{\bar{a}}$ and $j^{\bar{a}\bar{b}}$ are the linear and angular momentum, respectively, of the spinning particle when measured about $\bar{\gamma}(\tau)$. These equations are precisely the transport law for $p^{\bar{a}}$ and $j^{\bar{a}\bar{b}}$ using dual Killing transport. However, note that $p^{\bar{a}}$ and $j^{\bar{a}\bar{b}}$ are both parallel-transported along the geodesics connecting the two worldlines during the measurement process, and when the observer compares her initial and final measurements. As such, this observable is not a holonomy, as different transport laws are used along different portions of the loop.

The second difference between this procedure and the holonomy is that the holonomy can be computed around an arbitrary loop, whereas in this procedure the worldlines γ and $\bar{\gamma}$ are more constrained. The curve $\bar{\gamma}$ here refers to a "reference worldline" for the spinning test particle. This is the center of mass worldline of the particle, and is fixed by certain *spin-supplementary conditions* (for a review, see [111] and the references therein). We will discuss our choice of spin-supplementary condition further in section 4.2.2.3, as it is crucial for determining the acceleration of the spinning particle and therefore the exact shape of the loop. Related to our choice of spin supplementary condition is our definition of the intrinsic spin per unit mass, which we discuss in the same section.

Explicitly, our observables are $\Delta\xi_{\mathrm{S}}^{a'}$, $\Delta p^{a'}$, and $\Delta s^{a'}$, which are (respectively) the differences between initial and final separation, measured linear momenta, and measured intrinsic spins per unit mass of the test particle. These are functions of the initially measured four-momentum p^a, initial intrinsic spin per unit mass s^a, and initial separation ξ^a. We expand both in the separation and in the intrinsic spin, assuming that the size of the body, which is characterized by the spin per unit mass, is small as well. That is, our approximation is that

$$|\boldsymbol{\xi}| \gtrsim |\boldsymbol{s}| \gg m, \tag{4.45}$$

where m is the mass of the particle. The assumption that the spin per unit mass is much larger than the mass is necessary in order to neglect the effects of self-force on the test particle; this condition holds for extended bodies such as the earth, but not for black holes.

To second order in separation, but first order in intrinsic spin, we can write

$$\Delta\xi_{\mathrm{S}}^{a'} \equiv \left[\Delta K^{a'}{}_b + L^{a'}{}_{bc}\xi^c + O(\boldsymbol{\xi}^2)\right]\xi^b + \left[\Upsilon^{a'}{}_b + \Psi^{a'}{}_{bc}\xi^c + O(\boldsymbol{\xi}^2)\right] s^b + O(\boldsymbol{s})^2, \tag{4.46a}$$

$$\Delta p^{a'} = m\frac{\mathrm{D}\Delta\xi_{\mathrm{S}}^{a'}}{\mathrm{d}\tau_1} + O(\boldsymbol{s})^2, \tag{4.46b}$$

$$\Delta s^{a'} \equiv -\left[{}_\gamma g^{a'}{}_a\Omega^a{}_b(\gamma,\bar{\gamma};\tau_1) + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2\right] s^b + O(\boldsymbol{s}^2). \tag{4.46c}$$

The first of these expressions can be considered as definitions of $\Upsilon^{a'}{}_b$ and $\Psi^{a'}{}_{bc}$. These are bitensors determined by the curve γ whose explicit forms will be calculated in section 4.2.3.3. The quantities $\Delta K^{a'}{}_b$ and $L^{a'}{}_{bc}$ are the same as those introduced in section 4.1.2, and their values are given in section 4.2.3.1. Equations (4.46b) and (4.46c) are results that will be proven in section 4.2.3.3.

Our final expressions for these observables are given in equation (4.177); assuming weak curvature, we find that these expressions are given by

$$\Upsilon^\alpha{}_\beta = -\int_{\tau_0}^{\tau_1}\mathrm{d}\tau_2\int_{\tau_0}^{\tau_2}\mathrm{d}\tau_3(R^*)^\alpha{}_{\gamma\beta\delta}(\tau_3)\dot{\gamma}^\gamma\dot{\gamma}^\delta + O(\boldsymbol{R}^2), \tag{4.47a}$$

$$\Psi^\alpha{}_{\beta\gamma} = -\int_{\tau_0}^{\tau_1}\mathrm{d}\tau_2\int_{\tau_0}^{\tau_2}\mathrm{d}\tau_3[\nabla_\gamma(R^*)^\alpha{}_{\delta\beta\epsilon}](\tau_3)\dot{\gamma}^\delta\dot{\gamma}^\epsilon + O(\boldsymbol{R}^2). \tag{4.47b}$$

These expressions are much simpler than the corresponding expressions for the holonomy, or for curve deviation. This is primarily because the motion of the spinning body is already specified, and so the relative velocity and accelerations take on very particular forms.

4.1.5 Feasibility of measurement

All of the persistent observables in this section are (in principle) measurable by *some* detector, by design. For curve deviation and our observable involving a spinning test particle, the procedures involved in their definitions are relatively straightforward to perform. The former requires that the observers be able to measure separations and to keep track of their respective accelerations, while the latter requires a method by which an observer can measure the momentum and spin of a particle, in addition to separation. The holonomy observable is much more complex, as it requires the two observers to measure the local curvature of spacetime, potentially by carrying around miniature gravitational wave detectors themselves. The observers could then use the measured curvature to evolve the quantities P^a and J^{ab} according to equation (4.19), and finally compare their results at the end.

A far simpler method that one could use to measure these observables, without constructing new types of detectors, would be to take advantage of the fact that the values of these persistent observables can be written, in the weak curvature limit, in terms of integrals of the Riemann tensor (and its derivatives) along the worldline of one of the observers. This weak curvature limit is expected to be valid for observers far from an astrophysical source of gravitational waves. Moreover, when the observer is far enough from the source, the gravitational waves can be well approximated by plane waves. This implies that the derivatives of the Riemann tensor can be expressed solely in terms of derivatives with respect to retarded time. Since retarded time is an affine parameter for the worldline of an observer at fixed radius, terms involving integrals of the derivatives of the Riemann tensor can be drastically simplified.

In this regime, and in the case where there is no acceleration, the weak curvature results in the preceding sections, that is, equations (4.10), (4.47), and (4.29) [when combined with equation (4.30)] involve only one, two, and three time integrals of the Riemann tensor along the worldline of the detector (allowing for acceleration terms, these results include higher time integrals). Gravitational wave detectors measure the components of the Riemann tensor, and these components can be integrated in time while the gravitational waves are passing by. As such, gravitational wave detectors could use these measurements and our weak curvature results to deduce the value of any of the persistent observables we have discussed in this section, had the detector been carrying out the operations by which these observables are defined.

Coincidentally, these numbers of time integrals of the Riemann tensor have appeared in previous discussions of persistent gravitational wave observables: one time integral for the relative proper time, velocity, and rotation observables [equations (4.6), (4.4), and (4.5), respectively], two for the displacement memory [equation (4.2)], and three for the subleading displacement memory [equation (4.3)]. As such, in the limit discussed in this section, the only new information contained in these observables is the higher time integrals of the Riemann tensor that arise in the holonomy and curve deviation when there are acceleration terms. In situations where this limit is not appropriate, the observables presented in this section are not expected to be degenerate with those previously discussed in the literature. However, as we will show in section 4.3.3, in the case of nonlinear plane waves, there are still degeneracies that are present.

4.2 Arbitrary Flat-to-Flat Transitions

In this section, we consider persistent observables in the context of arbitrary flat-to-flat transitions, deriving results using the theory of covariant bitensors. The results in this section are necessarily perturbative.

The structure of this section is as follows. We first review the formalism of covariant bitensors in section 4.2.1, and in section 4.2.2 we give various applications to computations of the holonomy, generalizations of geodesic deviation, and solving the Mathisson-Papapetrou equations (4.44). In section 4.2.3, we provide results for the specific persistent observables introduced in section 4.1.

4.2.1 Review of covariant bitensors

In this section, we provide a review of techniques that can be used in arbitrary spacetimes to derive results for persistent observables, particularly ones that are associated with flat-to-flat transitions.

To define a bitensor, consider two vector spaces V and V' at x and x', respectively. A tensor at x, for example, is simply defined by being a multilinear map from V and V^* (its dual space), to $\mathbb{R}$. However, one can also consider multilinear maps from V, V^*, V', and $(V')^*$ to $\mathbb{R}$—these are *bitensors*. A bitensor field is, then, simply an assignment of a bitensor to each pair of points in a manifold. In the usual abstract index notation, whereas a tensor is denoted by $T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}$, a bitensor has two sets of indices, corresponding to the action of the multilinear map on both sets of vector spaces and dual spaces; for example, $T^{a_1 \cdots a_r a'_1 \cdots a'_{r'}}{}_{b_1 \cdots b_s b'_1 \cdots b'_{s'}}$ represents a bitensor of rank (r, s) at x, and rank (r', s') at x'. Note that a bitensor field may be a scalar at some point, and in that case we will denote the dependence of such a bitensor field on that point explicitly, as there are no indices to indicate its scalar dependence.

With the basic idea of a bitensor introduced, in the rest of this section, we review examples of bitensors, such as Synge's world function and parallel and Jacobi propagators. We then review coincidence limits, which form the basis of many of the computations in the next section, where we give applications of covariant bitensors.

At this point, we make a brief remark about notation. Although we will primarily be considering only the tangent bundle and the linear and angular momentum bundle in this chapter (see section 4.1.3), most of what follows in this section can be applied to any connection on any vector bundle. As such, we will use $\hat{\nabla}_a$ and $\check{\nabla}_a$ as arbitrary connections. We will also use capital Latin letters for indices on a generic bundle, not just the linear and angular momentum bundle. Throughout this section, we add a diacritical mark above the core symbol of any tensor that depends on a given connection with the same diacritic above the connection, for example with the parallel propagator ${}_\gamma\hat{g}^{A'}{}_A$ [equation (4.53)], which is defined with respect to $\hat{\nabla}_a$. Furthermore, for quantities that depend on two connections, we add both diacritics above the core symbol for the diacritical marks associated with the two connections; for example, the connection coefficient $\hat{\check{C}}^A{}_{Bc}$ [equation (4.61)] is defined with respect to $\hat{\nabla}_a$ and $\check{\nabla}_a$.

4.2.1.1 Synge's world function

The first bitensor which we consider is, in fact, a *biscalar*: a scalar function of two points on a manifold. We start by considering a so-called *convex normal neighborhood*. This is a subset of a manifold small enough that, for all points x and x' in this neighborhood, there is a unique geodesic $\Gamma_{(x,x')}$ satisfying

$$\Gamma_{(x,x')}(0) = x, \qquad \Gamma_{(x,x')}(1) = x'. \tag{4.48}$$

In such a neighborhood, we can define *Synge's world function* $\sigma(x, x')$, as half of the squared distance along $\Gamma_{(x,x')}$:

$$\sigma(x, x') = \frac{1}{2}\int_0^1 \mathrm{d}\lambda\; g_{a''b''}\dot{\Gamma}^{a''}_{(x,x')}\dot{\Gamma}^{b''}_{(x,x')}, \tag{4.49}$$

where $x'' \equiv \Gamma_{(x,x')}(\lambda)$. The derivatives of Synge's world function are denoted by appending indices onto $\sigma(x, x')$:

$$\nabla_{a_1}\cdots\nabla_{a_n}\nabla_{a'_1}\cdots\nabla_{a'_n}\sigma(x, x') \equiv \sigma_{a'_n\cdots a'_1 a_n\cdots a_1}. \tag{4.50}$$

Covariant derivatives with respect to different points, of course, commute. Note that (as shown, for example, in [131])

$$\sigma^a(x') = -\dot{\Gamma}^a_{(x,x')}, \qquad \sigma^{a'}(x) = \dot{\Gamma}^{a'}_{(x,x')}. \tag{4.51}$$

This shall be our primary use for Synge's world function, since it provides a notion of separation vector between two closely-separated points.

4.2.1.2 Parallel propagators

In this section we define the *parallel propagators* ${}_\gamma\hat{g}^{A'}{}_A$ and ${}_\gamma\hat{g}^{A}{}_{A'}$, which are bitensors at $x \equiv \gamma(\tau)$ and $x' \equiv \gamma(\tau')$, and are defined with respect to a connection $\hat{\nabla}_a$ on some arbitrary vector bundle on our manifold. We then find expressions to relate parallel propagators that are defined with respect two different connections $\hat{\nabla}_a$ and $\check{\nabla}_a$.

To construct the parallel propagator on an arbitrary d-dimensional bundle, consider a basis of vectors at $x = \gamma(\tau)$ denoted by $\{(\hat{e}_\Gamma)^A \mid \Gamma = 1, \ldots, d\}$ and the basis of one-forms dual to this basis denoted by $\{(\hat{\omega}^\Gamma)_A \mid \Gamma = 1, \ldots, d\}$. Now, parallel transport $(\hat{e}_\Gamma)^A$ along γ from $\gamma(\tau)$ to $x' = \gamma(\tau')$, with respect to the connection $\hat{\nabla}_a$, to yield $(\hat{e}_\Gamma)^{A'}$, with dual basis $(\hat{\omega}^\Gamma)_{A'}$:

$$\frac{\hat{\mathrm{D}}(\hat{e}_\Gamma)^{A'}}{\mathrm{d}\tau'} = 0, \tag{4.52}$$

where $\hat{\mathrm{D}}/\mathrm{d}\tau' \equiv \dot{\gamma}^{a'}\hat{\nabla}_{a'}$. From these tetrads, we can define the parallel propagators by

$${}_\gamma\hat{g}^{A'}{}_A \equiv \sum_{\Gamma=1}^{d} (\hat{e}_\Gamma)^{A'}(\hat{\omega}^\Gamma)_A, \qquad {}_\gamma\hat{g}^{A}{}_{A'} \equiv \sum_{\Gamma=1}^{d} (\hat{e}_\Gamma)^{A}(\hat{\omega}^\Gamma)_{A'}. \tag{4.53}$$

Note that this is not the usual parallel propagator defined in, say, [131], as the bases are parallel-transported along a specific curve. This is the significance of the subscripted γ that is added to the left of the g. Moreover, this definition allows for connections that are not metric compatible, and does not require these bases to be either orthogonal or normalized with respect to any metric.

As the bases were parallel-transported with respect to $\hat{\nabla}_a$, the parallel propagators satisfy

$$\frac{\hat{\mathrm{D}}}{\mathrm{d}\tau}\,{}_\gamma\hat{g}^{A'}{}_A = 0, \qquad \frac{\hat{\mathrm{D}}}{\mathrm{d}\tau'}\,{}_\gamma\hat{g}^{A'}{}_A = 0. \tag{4.54}$$

A similar result holds for ${}_\gamma\hat{g}^{A}{}_{A'}$. These equations can be considered to be the *definitions* of the parallel propagators. Note that this means that

$$X^{A'}(\tau) \equiv {}_\gamma\hat{g}^{A'}{}_A X^A, \qquad Y_{A'}(\tau) \equiv Y_A\,{}_\gamma\hat{g}^{A}{}_{A'} \tag{4.55}$$

are the unique solutions to the differential equations

$$\frac{\hat{\mathrm{D}}}{\mathrm{d}\tau'} X^{A'}(\tau) = 0, \qquad \frac{\hat{\mathrm{D}}}{\mathrm{d}\tau'} Y_{A'}(\tau) = 0, \tag{4.56}$$

with initial values X^A and Y_A, respectively. Moreover, one can show that

$$ {}_\gamma\hat{g}^{A}{}_{A'}\, {}_\gamma\hat{g}^{A'}{}_{B} = \delta^{A}{}_{B}. \tag{4.57} $$

as well as

$$ {}_\gamma\hat{g}^{A''}{}_{A'}\, {}_\gamma\hat{g}^{A'}{}_{A} = {}_\gamma\hat{g}^{A''}{}_{A}. \tag{4.58} $$

We remarked earlier that this parallel propagator is defined with respect to a specific curve γ. This specification can be relaxed in a convex normal neighborhood, yielding the usual parallel propagator $\hat{g}^{A'}{}_{A}$, which is defined by

$$ \hat{g}^{A'}{}_{A} \equiv {}_{\Gamma(x,x')}\hat{g}^{A'}{}_{A}. \tag{4.59} $$

This bitensor satisfies

$$ \sigma^{b}(x')\hat{\nabla}_{b}\hat{g}^{A'}{}_{A} = 0, \qquad \sigma^{b'}(x)\hat{\nabla}_{b'}\hat{g}^{A'}{}_{A} = 0, \tag{4.60} $$

with again a similar result holding for $\hat{g}^{A}{}_{A'}$. Note that, even when γ is geodesic, we still prefer ${}_\gamma\hat{g}^{A'}{}_{A}$ to $\hat{g}^{A'}{}_{A}$, since γ may not be the only geodesic between x and x', when $\tau' - \tau$ is sufficiently large. Therefore, unless we are considering the parallel propagators between two points that are *defined* to be always within a convex normal neighborhood, we will explicitly specify the curve γ.

Next, we consider the case where we have two connections, $\hat{\nabla}_a$ and $\check{\nabla}_a$, defined on this vector bundle. The connection coefficient $\hat{\check{C}}{}^{A}{}_{Bc}$ is defined by

$$ (\hat{\nabla}_b - \check{\nabla}_b)X^A \equiv \hat{\check{C}}{}^{A}{}_{Cb}X^C, \tag{4.61} $$

where clearly $\hat{\check{C}}{}^{A}{}_{Bc} = -\check{\hat{C}}{}^{A}{}_{Bc}$. Note that ${}_\gamma\hat{g}^{A'}{}_{A}$ satisfies

$$ \frac{\mathrm{d}}{\mathrm{d}\tau'}\left({}_\gamma\check{g}^{A}{}_{A'}\, {}_\gamma\hat{g}^{A'}{}_{B}\right) = -\, {}_\gamma\check{g}^{A}{}_{A'}\hat{\check{C}}{}^{A'}{}_{C'd'}\dot{\gamma}^{d'}\, {}_\gamma\hat{g}^{C'}{}_{B}. \tag{4.62} $$

Thus, ${}_\gamma\hat{g}^{A'}{}_{A}$ is a solution to the following integral equation:

$$ {}_\gamma\hat{g}^{A'}{}_{B} = {}_\gamma\check{g}^{A'}{}_{A}\left(\delta^{A}{}_{B} - \int_{\tau}^{\tau'} \mathrm{d}\tau''\, {}_\gamma\hat{\check{A}}{}^{A}{}_{B''}\, {}_\gamma\hat{g}^{B''}{}_{B}\right), \tag{4.63} $$

where $x'' \equiv \gamma(\tau'')$ and

$$ {}_\gamma\hat{\check{A}}{}^{A}{}_{B'} \equiv {}_\gamma\check{g}^{A}{}_{A'}\hat{\check{C}}{}^{A'}{}_{B'c'}\dot{\gamma}^{c'}. \tag{4.64} $$

By the same logic, one can show that

$$ {}_\gamma\hat{g}^{A}{}_{B'} = {}_\gamma\breve{g}^{A}{}_{A'} \left(\delta^{A'}{}_{B'} + \int_\tau^{\tau'} \mathrm{d}\tau''\, {}_\gamma\hat{\mathring{A}}^{A'}{}_{B''}\, {}_\gamma\hat{g}^{B''}{}_{B'} \right), \tag{4.65} $$

We typically solve equations (4.63) and (4.65) iteratively, either by truncating the expansion based on a particular approximation scheme, or by exploiting the fact that (for certain connections) ${}_\gamma\hat{\mathring{A}}^{A}{}_{B'}$ is nilpotent.

4.2.1.3 Jacobi propagators

As remarked in the previous section, the parallel propagators provide a way to define solutions to the differential equation

$$ \frac{\hat{\mathrm{D}}}{\mathrm{d}\tau'} X^{A'} = 0, \tag{4.66} $$

with the initial value X^A. In this section, we consider a different differential equation, which we will also be solving for given initial values ξ^a and $\mathrm{D}\xi^a/\mathrm{d}\tau$:

$$ \frac{\mathrm{D}^2}{\mathrm{d}\tau'^2} \xi^{a'} = -R^{a'}{}_{c'b'd'} \dot{\gamma}^{c'} \dot{\gamma}^{d'} \xi^{b'}. \tag{4.67} $$

To construct the solutions to this differential equation, start with two sets of bases $\{({}_K e_\alpha)^a \mid \alpha = 0, \dots, 3\}$ and $\{({}_H e_\alpha)^a \mid \alpha = 0, \dots, 3\}$ at $x \equiv \gamma(\tau)$, and propagate them along γ by supposing that, at all $x' \equiv \gamma(\tau')$,

$$ \frac{\mathrm{D}^2}{\mathrm{d}\tau'^2} ({}_K e_\alpha)^{a'} = -R^{a'}{}_{c'b'd'} \dot{\gamma}^{c'} \dot{\gamma}^{d'} ({}_K e_\alpha)^{b'}, \tag{4.68a} $$

$$ \frac{\mathrm{D}^2}{\mathrm{d}\tau'^2} \left[(\tau' - \tau)({}_H e_\alpha)^{a'} \right] = -(\tau' - \tau) R^{a'}{}_{c'b'd'} \dot{\gamma}^{c'} \dot{\gamma}^{d'} ({}_H e_\alpha)^{b'}. \tag{4.68b} $$

At each τ', there exist dual bases $\{({}_K \omega^\alpha)_{a'} \mid \alpha = 0, \dots, 3\}$ and $\{({}_H \omega^\alpha)_{a'} \mid \alpha = 0, \dots, 3\}$, and we use them to construct the *Jacobi propagators* ${}_\gamma K^{a'}{}_a$ and ${}_\gamma H^{a'}{}_a$:

$$ {}_\gamma K^{a'}{}_a \equiv \sum_{\alpha=0}^{3} ({}_K e_\alpha)^{a'} ({}_K \omega^\alpha)_a, \qquad {}_\gamma H^{a'}{}_a \equiv \sum_{\alpha=0}^{3} ({}_H e_\alpha)^{a'} ({}_H \omega^\alpha)_a. \tag{4.69} $$

In terms of the Jacobi propagators,

$$ \xi^{a'} = {}_\gamma K^{a'}{}_a \xi^a + (\tau_1 - \tau_0)\, {}_\gamma H^{a'}{}_a \frac{\mathrm{D}\xi^a}{\mathrm{d}\tau}. \tag{4.70} $$

It is conventional not to absorb the factor of $\tau_1 - \tau_0$ into the definition of ${}_\gamma H^{a'}{}_a$, as it is convenient for defining the Jacobi propagators in terms of Synge's world function [173].

Much like ${}_\gamma\hat{g}^{A'}{}_A$, one can show that ${}_\gamma K^{a'}{}_a$ and $(\tau'-\tau)\,{}_\gamma H^{a'}{}_a$ are both solutions $A^{a'}{}_a$ to the differential equation

$$\frac{\mathrm{D}^2}{\mathrm{d}\tau'^2}A^{a'}{}_a = -R^{a'}{}_{c'b'd'}\dot{\gamma}^{c'}\dot{\gamma}^{d'}A^{b'}{}_a, \tag{4.71}$$

with the initial conditions

$${}_\gamma K^a{}_b = {}_\gamma H^a{}_b = \delta^a{}_b, \tag{4.72a}$$

$$\lim_{\tau'\to\tau}\left({}_\gamma g^a{}_{a'}\frac{\mathrm{D}\,{}_\gamma K^{a'}{}_b}{\mathrm{d}\tau'}\right) = \lim_{\tau'\to\tau}\left({}_\gamma g^a{}_{a'}\frac{\mathrm{D}\,{}_\gamma H^{a'}{}_b}{\mathrm{d}\tau'}\right) = 0. \tag{4.72b}$$

This is an equivalent way of defining the Jacobi propagators.

We now wish to solve a slightly different differential equation,

$$\frac{\mathrm{D}^2\xi^{a'}}{\mathrm{d}\tau^2} = -R^{a'}{}_{c'b'd'}\dot{\gamma}^{c'}\dot{\gamma}^{d'}\xi^{b'} + S^{a'}. \tag{4.73}$$

To do so, it is useful to consider equation (4.67) as a differential operator, which we denote by $\mathcal{J}_\tau$, acting on a vector X^A that exists in a bundle that is a direct sum of two copies of the tangent space, which we call the *Jacobi bundle*:

$$X^A \equiv \begin{pmatrix}\xi^a\\ \tilde{\xi}^a\end{pmatrix}. \tag{4.74}$$

Equation (4.67) can be written as

$$\mathcal{J}_\tau X^A \equiv \begin{pmatrix}\frac{\mathrm{D}\xi^a}{\mathrm{d}\tau} - \tilde{\xi}^a\\ \frac{\mathrm{D}\tilde{\xi}^a}{\mathrm{d}\tau} + R^a{}_{cbd}\dot{\gamma}^c\dot{\gamma}^d\xi^a\end{pmatrix} = 0. \tag{4.75}$$

It follows that equation (4.73) becomes

$$\mathcal{J}_\tau X^A = S^A, \tag{4.76}$$

where

$$S^A \equiv \begin{pmatrix}0\\ S^a\end{pmatrix}. \tag{4.77}$$

At this point, define ${}_\gamma W^{A'}{}_A$ by

$${}_\gamma W^{A'}{}_A \equiv \begin{pmatrix}{}_\gamma K^{a'}{}_a & (\tau'-\tau)\,{}_\gamma H^{a'}{}_a\\ \frac{\mathrm{D}\,{}_\gamma K^{a'}{}_a}{\mathrm{d}\tau'} & \frac{\mathrm{D}[(\tau'-\tau)\,{}_\gamma H^{a'}{}_a]}{\mathrm{d}\tau'}\end{pmatrix}. \tag{4.78}$$

This is matrix on the Jacobi bundle that clearly satisfies

$$\mathcal{J}_{\tau'}\, {}_\gamma W^{A'}{}_A = 0 \tag{4.79}$$

and

$${}_\gamma W^{A''}{}_{A'}\, {}_\gamma W^{A'}{}_A = {}_\gamma W^{A''}{}_A. \tag{4.80}$$

The latter of these two equations implies that

$${}_\gamma W^{A}{}_{A'}\, {}_\gamma W^{A'}{}_B = \delta^A{}_B, \tag{4.81}$$

from which the former implies that, assuming that $\mathcal{J}_\tau$ is extended to covectors in the Jacobi bundle such that the Leibniz rule holds,

$$\mathcal{J}_{\tau}\, {}_\gamma W^{A'}{}_A = 0. \tag{4.82}$$

To solve equation (4.73), note that by equations (4.82) and (4.76),

$$\frac{\mathrm{d}}{\mathrm{d}\tau'}\left({}_\gamma W^{A}{}_{A'} X^{A'}\right) = {}_\gamma W^{A}{}_{A'} S^{A'}, \tag{4.83}$$

and so by an integration and equations (4.81) and (4.80), it follows that

$$X^{A'} = {}_\gamma W^{A'}{}_A X^A + \int_\tau^{\tau'} \mathrm{d}\tau''\, {}_\gamma W^{A'}{}_{A''} S^{A''}. \tag{4.84}$$

This implies that

$$\xi^{a'} = {}_\gamma K^{a'}{}_a \xi^a + (\tau' - \tau)\, {}_\gamma H^{a'}{}_a \frac{\mathrm{D}\xi^a}{\mathrm{d}\tau} + \int_\tau^{\tau'} \mathrm{d}\tau'' (\tau' - \tau'')\, {}_\gamma H^{a'}{}_{a''} S^{a''}. \tag{4.85}$$

As equation (4.73) arises in the study of generalizations of geodesic deviation, its solution in equation (4.85) is quite important.

There are a few additional identities that are useful in studying the Jacobi propagators (see, for example, [88]). The first is that, for any solutions $A^{a'}{}_a$ and $B^{a'}{}_a$ to equation (4.71), one can show the following:

$$\frac{\mathrm{D}}{\mathrm{d}\tau'}\left[g_{a'b'}\left(B^{b'}{}_b \frac{\mathrm{D}A^{a'}{}_a}{\mathrm{d}\tau'} - A^{a'}{}_a \frac{\mathrm{D}B^{b'}{}_b}{\mathrm{d}\tau'}\right)\right] = 0. \tag{4.86}$$

Setting $A^{a'}{}_a = (\tau' - \tau)\, {}_\gamma H^{a'}{}_a$ and $B^{a'}{}_a = (\tau' - \tau'')\, {}_\gamma H^{a'}{}_{a''}\, {}_\gamma g^{a''}{}_a$, one can show that

$${}_\gamma H^{a'}{}_a = g_{ab} g^{a'b'}\, {}_\gamma H^b{}_{b'}. \tag{4.87}$$

Similarly, using $A^{a'}{}_a = {}_\gamma K^{a'}{}_a$ and $B^{a'}{}_a = {}_\gamma K^{a'}{}_{a''}\, {}_\gamma g^{a''}{}_a$, one can show that

$$\frac{\mathrm{D}\, {}_\gamma K^{a'}{}_a}{\mathrm{d}\tau'} = -g_{ab} g^{a'b'} \frac{\mathrm{D}\, {}_\gamma K^a{}_{a'}}{\mathrm{d}\tau'}. \tag{4.88}$$

Another useful identity comes from noting that $\mathrm{D}\, {}_\gamma K^{a'}{}_a/\mathrm{d}\tau$ and $\mathrm{D}[(\tau' - \tau)\, {}_\gamma H^{a'}{}_a]/\mathrm{d}\tau$ obey equation (4.71); by comparing boundary conditions at $\tau = \tau'$, one can show that

$$\frac{\mathrm{D}\, {}_\gamma K^{a'}{}_a}{\mathrm{d}\tau} = (\tau' - \tau)\, {}_\gamma H^{a'}{}_b R^b{}_{cad} \dot\gamma^c \dot\gamma^d, \qquad \frac{\mathrm{D}[(\tau' - \tau)\, {}_\gamma H^{a'}{}_a]}{\mathrm{d}\tau} = -\, {}_\gamma K^{a'}{}_a. \tag{4.89}$$

Finally, as with the parallel propagator, one can define versions of the Jacobi propagators in a convex normal neighborhood:

$$K^{a'}{}_a \equiv {}_{\Gamma(x,x')} K^{a'}{}_a, \qquad H^{a'}{}_a \equiv {}_{\Gamma(x,x')} H^{a'}{}_a. \tag{4.90}$$

These bitensors, as it turns out, are related to derivatives of Synge's world function (see, for example, [173]):

$$H^{a'}{}_b \sigma^b{}_{b'} = -\delta^{a'}{}_{b'}, \qquad K^{a'}{}_b = H^{a'}{}_a \sigma^a{}_b(x'). \tag{4.91}$$

4.2.1.4 Coincidence limits

In this section, we briefly review coincidence limits and give expressions for the coincidence limits that we will need for the rest of this chapter. The *coincidence limit* of a bitensor $T_{A_1 \cdots A_r B'_1 \cdots B'_s}$ is given by

$$\left[T_{A_1 \cdots A_r B'_1 \cdots B'_s}\right]_{x' \to x} \equiv \lim_{x' \to x} \hat g^{B'_1}{}_{B_1} \cdots \hat g^{B'_s}{}_{B_s} T_{A_1 \cdots A_r B'_1 \cdots B'_s}, \tag{4.92}$$

using the parallel propagator with respect to the connection $\hat\nabla_a$; it is trivial to show that this is independent of the particular connection used, which is why there is no dependence on $\hat\nabla_a$ on the left-hand side. By convention, the indices inside the coincidence limit that are associated with the point whose limit is being taken (in this case x') are treated as if they were at the limiting point (in this case x) for expressions outside of the brackets. We use this notation throughout, following Poisson's review article [131]; simple examples can be seen below in equations (4.93) and (4.94).

We now list the coincidence limits which we will need in the rest of this chapter. A general procedure for computing these coincidence limits is outlined in [131]. These expressions can also be found in [174]. For Synge's world function, the relevant coincidence limits are

$$\delta^a{}_b = [\sigma^a{}_b]_{x'\to x} = -\,[\sigma^a{}_{b'}]_{x'\to x}\,, \tag{4.93a}$$

$$0 = [\sigma^a{}_{bc'}]_{x'\to x} = [\sigma^a{}_{b'c'}]_{x'\to x}\,, \tag{4.93b}$$

$$-\frac{2}{3}R^a{}_{(c|b|d)} = [\sigma^a{}_{bc'd'}]_{x'\to x} = 2\left[\sigma^a{}_{b'(c'd')}\right]_{x'\to x}\,, \tag{4.93c}$$

while for the parallel propagator, they are

$$0 = \left[\hat{\nabla}_c \hat{g}^{A'}{}_B\right]_{x'\to x} = \left[\hat{\nabla}_{c'} \hat{g}^{A'}{}_B\right]_{x'\to x}\,, \tag{4.94a}$$

$$\begin{aligned}\frac{1}{2}\hat{R}^A{}_{Bcd} &= \left[\hat{\nabla}_{c'}\hat{\nabla}_{d'} \hat{g}^{A'}{}_B\right]_{x'\to x} = \left[\nabla_{c'}\nabla_{d} g^{A'}{}_B\right]_{x'\to x} = -\left[\hat{\nabla}_{c}\hat{\nabla}_{d'} \hat{g}^{A'}{}_B\right]_{x'\to x} \\ &= -\left[\hat{\nabla}_{c}\hat{\nabla}_{d} \hat{g}^{A'}{}_B\right]_{x'\to x}\,,\end{aligned} \tag{4.94b}$$

$$\frac{2}{3}\hat{\nabla}_{(c}\hat{R}^A{}_{|B|d)e} = \left[\nabla_{c'}\nabla_{d'}\nabla_{e'} \hat{g}^{A'}{}_B\right]_{x'\to x} = 2\left[\nabla_{(c'}\nabla_{d')}\nabla_{e} \hat{g}^{A'}{}_B\right]_{x'\to x}\,, \tag{4.94c}$$

where $\hat{R}^A{}_{Bcd}$ is the curvature tensor defined with respect to the connection $\hat{\nabla}_a$ and is defined by

$$2\hat{\nabla}_{[c}\hat{\nabla}_{d]}X^A \equiv \hat{R}^A{}_{Bcd}X^B. \tag{4.95}$$

For two connections $\hat{\nabla}_a$ and $\check{\nabla}_a$, their curvature tensors are related by

$$\hat{R}^A{}_{Bcd} = \check{R}^A{}_{Bcd} + 2\check{\nabla}_{[c}\hat{\check{C}}^A{}_{|B|d]} + 2\hat{\check{C}}^A{}_{E[c}\hat{\check{C}}^E{}_{|B|d]}. \tag{4.96}$$

Moreover, for any bitensor $T_{A_1\cdots A_r B'_1\cdots B'_s}$ [174],

$$\left[T_{A_1\cdots A_r B'_1\cdots B'_s}\right]_{x'\to x} = \left[T_{A'_1\cdots A'_r B_1\cdots B_s}\right]_{x'\to x}. \tag{4.97}$$

All of the coincidence limits which are needed in this chapter can be derived from using this property of coincidence limits, along with equations (4.93) and (4.94).

4.2.2 Applications of bitensors

We now turn to applications of the theory of covariant bitensors in three areas that are crucial for deriving results about persistent observables: geodesic deviation and its generalizations, the holonomies of transport laws, and the solutions to the Mathisson-Papapetrou equations.

4.2.2.1 Non-geodesic deviation

In this section, we review the computation of the separation vector in terms of the initial separation and its derivative, as well as the accelerations of the worldlines. This forms the basis of the curve deviation observable introduced in section 4.1.2. By equation (4.122), this is also necessary to calculate the holonomy, as well as the persistent observable involving a spinning particle (as is apparent from the definition in section 4.1.4). We carry out this calculation to second order in ξ^a and $\dot{\xi}^a$.

We start with considering by two closely-separated, timelike, and affinely parametrized curves γ and $\bar{\gamma}$. In terms of Synge's world function, the separation vector ξ^a is given by

$$\xi^a \equiv -\sigma^a[\bar{\gamma}(\tau)], \tag{4.98}$$

for any τ. As such, we are explicitly using the isochronous correspondence mentioned in section 4.1.2 above, where the separation vector connects points with equal values of affine parameter. For convenience, we assume that this shared affine parameter is the proper time of both worldlines, and thus is fixed up to an additive constant, which can be set initially by requiring $\xi^a \dot{\gamma}_a = 0$ (for example).

We further define the relative velocity

$$\dot{\xi}^a \equiv \frac{\mathrm{D}\xi^a}{\mathrm{d}\tau} = \left(\dot{\gamma}^b \nabla_b + \dot{\bar{\gamma}}^{\bar{b}} \nabla_{\bar{b}}\right) \xi^a, \tag{4.99}$$

where the second equality follows from the fact that ξ^a is a function of τ due to its dependence on both $\gamma(\tau)$ and $\bar{\gamma}(\tau)$. As such, using equation (4.93), we find that

$$\dot{\xi}^a = g^a{}_{\bar{a}} \dot{\bar{\gamma}}^{\bar{a}} - \dot{\gamma}^a + \frac{1}{6} R^a{}_{cbd} \xi^c \xi^d (g^b{}_{\bar{b}} \dot{\bar{\gamma}}^{\bar{b}} + 2\dot{\gamma}^b) + O(\boldsymbol{\xi}^3). \tag{4.100}$$

One can iteratively invert this equation order by order, yielding

$$\dot{\bar{\gamma}}^{\bar{a}} = g^{\bar{a}}{}_a \left[\dot{\gamma}^a + \dot{\xi}^a - \frac{1}{6} R^a{}_{bcd} \xi^b \left(3\dot{\gamma}^c + \dot{\xi}^c\right) \xi^d\right] + O(\boldsymbol{\xi}^3). \tag{4.101}$$

This equation will prove useful in section 4.2.2.2 as well.

We now take another derivative of equation (4.101) with respect to τ:

$$\begin{aligned} \ddot{\bar{\gamma}}^{\bar{a}} = &\left(\dot{\gamma}^a + \dot{\xi}^a - \frac{1}{2}R^a{}_{bcd}\xi^b\dot{\gamma}^c\xi^d\right)\left(\dot{\gamma}^e\nabla_e + \dot{\bar{\gamma}}^{\bar{e}}\nabla_{\bar{e}}\right)g^{\bar{a}}{}_a \\ &+ g^{\bar{a}}{}_a\left[\left(\delta^a{}_b - \frac{1}{2}R^a{}_{cbd}\xi^c\xi^d\right)\ddot{\gamma}^b + \left(\delta^a{}_b - \frac{1}{6}R^a{}_{cbd}\xi^c\xi^d\right)\ddot{\xi}^b \right. \\ &\left. - \frac{1}{2}\left(\xi^d\dot{\gamma}^e\nabla_e R^a{}_{bcd} + 2\dot{\xi}^d R^a{}_{(b|c|d)}\right)\xi^b\dot{\gamma}^c\right] + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^3. \end{aligned} \tag{4.102}$$

Using the coincidence limits of derivatives of the parallel propagators in equation (4.94), we find that

$$\left(\dot{\gamma}^b\nabla_b + \dot{\bar{\gamma}}^{\bar{b}}\nabla_{\bar{b}}\right)g^{\bar{a}}{}_a = g^{\bar{a}}{}_c\left[R^c{}_{abd}\left(\dot{\gamma}^d + \frac{1}{2}\dot{\xi}^d\right) + \frac{1}{2}\dot{\gamma}^d\xi^e\nabla_e R^c{}_{abd}\right]\xi^b + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^3. \tag{4.103}$$

Thus, equation (4.102) can be written as

$$\begin{aligned} g^a{}_{\bar{a}}\ddot{\bar{\gamma}}^{\bar{a}} = &\left(\delta^a{}_b - \frac{1}{2}R^a{}_{cbd}\xi^c\xi^d\right)\ddot{\gamma}^b + \left(\delta^a{}_b - \frac{1}{6}R^a{}_{cbd}\xi^c\xi^d\right)\ddot{\xi}^b \\ &+ \left[R^a{}_{cbd}\left(\dot{\gamma}^c\dot{\gamma}^d + \frac{1}{2}\dot{\gamma}^c\dot{\xi}^d + \dot{\xi}^c\dot{\gamma}^d\right) - \dot{\gamma}^c\dot{\xi}^d R^a{}_{(b|c|d)} + \frac{1}{2}\dot{\gamma}^c\dot{\gamma}^d\xi^e\nabla_e R^a{}_{cbd} - \frac{1}{2}\dot{\gamma}^c\xi^d\dot{\gamma}^e\nabla_e R^a{}_{bcd}\right]\xi^b \\ &+ O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^3. \end{aligned} \tag{4.104}$$

This equation can be solved for $\ddot{\xi}^a$, and then simplified using the (algebraic) Bianchi identity:

$$\begin{aligned} \ddot{\xi}^a = &- R^a{}_{cbd}\dot{\gamma}^c\dot{\gamma}^d\xi^b - 2R^a{}_{cbd}\xi^b\dot{\xi}^c\dot{\gamma}^d - \nabla_{(e}R^a{}_{c)bd}\xi^b\xi^c\dot{\gamma}^d\dot{\gamma}^e \\ &+ \left(\delta^a{}_b + \frac{1}{6}R^a{}_{cbd}\xi^c\xi^d\right)g^b{}_{\bar{b}}\ddot{\bar{\gamma}}^{\bar{b}} - \left(\delta^a{}_b - \frac{1}{3}R^a{}_{cbd}\xi^c\xi^d\right)\ddot{\gamma}^b + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^3. \end{aligned} \tag{4.105}$$

We now solve equation (4.105) order by order in ξ^a and $\dot{\xi}^a$. The linear solution, neglecting acceleration terms, is given simply by equation (4.70), and so we find that

$$\xi^{a'} = {}_\gamma K^{a'}{}_a\xi^a + (\tau' - \tau)\,{}_\gamma H^{a'}{}_a\dot{\xi}^a + O(\ddot{\gamma},\ddot{\bar{\gamma}}) + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2. \tag{4.106}$$

Now, insert this linear solution into the second-order equation (4.105); this equation then becomes an inhomogeneous, linear differential equation with a source term:

$$\ddot{\xi}^{a'} = -R^{a'}{}_{c'b'd'}\dot{\gamma}^{c'}\dot{\gamma}^{d'}\xi^{b'} + S^{a'}[\boldsymbol{\xi},\dot{\boldsymbol{\xi}},\ddot{\gamma},\ddot{\bar{\gamma}}] + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^3. \tag{4.107}$$

where $S^{a'}[\boldsymbol{\xi},\dot{\boldsymbol{\xi}},\ddot{\gamma},\ddot{\bar{\gamma}}]$ is a function of the initial ξ^a and $\dot{\xi}^a$, as well as the accelerations $\ddot{\gamma}^a$ and $\ddot{\bar{\gamma}}^{\bar{a}}$. Using equation (4.85), we therefore find that

$$\xi^{a'} = {}_\gamma K^{a'}{}_a\xi^a + (\tau' - \tau)\,{}_\gamma H^{a'}{}_a\dot{\xi}^a + \int_\tau^{\tau'} \mathrm{d}\tau''(\tau' - \tau'')\,{}_\gamma H^{a'}{}_{a''}S^{a''}[\boldsymbol{\xi},\dot{\boldsymbol{\xi}},\ddot{\gamma},\ddot{\bar{\gamma}}] + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^3. \tag{4.108}$$

4.2.2.2 Holonomies of transport laws

In terms of bitensors, the holonomy of a connection $\hat{\nabla}_a$ around some closed curve $\mathcal{C}$ is given by

$$ {}_{\mathcal{C}}\hat{\Lambda}^A{}_B = {}_{\mathcal{C}}\hat{g}^A{}_B. \tag{4.109} $$

If the closed curve is only piecewise smooth, composed of smooth paths $\mathcal{P}_1, \dots, \mathcal{P}_n$ with end points $x', \dots, x^{(n)}$, we instead write

$$ {}_{\mathcal{C}}\hat{\Lambda}^A{}_B = {}_{\mathcal{P}_n}\hat{g}^A{}_{B^{(n)}} \cdots {}_{\mathcal{P}_1}\hat{g}^{B'}{}_B \tag{4.110} $$

In this section, we will find expressions for the holonomies for various shapes, for an arbitrary connection $\hat{\nabla}_a$. We follow [174] in terms of general ideas, but the shapes that we consider are not assumed to be composed of geodesic segments, nor do we limit our results to the metric-compatible connection on the tangent bundle.

First, we show that the holonomy around a (nongeodesic) triangle is given by expressions involving the Riemann tensor associated with the connection on the vector bundle. Explicitly, consider the triangle depicted in figure 4.4, where two edges are segments of arbitrary curves γ and $\bar{\gamma}$ that meet at a point $x \equiv \gamma(0) \equiv \bar{\gamma}(0)$. Join $x' \equiv \gamma(\epsilon)$ and $\bar{x}' \equiv \bar{\gamma}(\bar{\epsilon})$ by the unique geodesic between them.

The holonomy around this triangle is given by

$$ {}_{\triangle}\hat{\Lambda}^A{}_B(\gamma, \bar{\gamma}; \epsilon, \bar{\epsilon}) \equiv {}_{\gamma}\hat{g}^A{}_{\bar{A}'}\hat{g}^{\bar{A}'}{}_{B'}\, {}_{\gamma}\hat{g}^{B'}{}_B. \tag{4.111} $$

Expanding the holonomy (which is assumed to be smooth in ϵ and $\bar{\epsilon}$) in a Taylor series, one finds that

$$ \begin{aligned} {}_{\triangle}\hat{\Lambda}^A{}_B(\gamma, \bar{\gamma}; \epsilon, \bar{\epsilon}) &= \sum_{m,n=0}^{\infty} \frac{\epsilon^m \bar{\epsilon}^n}{m!n!} \lim_{\epsilon,\bar{\epsilon}\to 0} \left[\frac{\partial^{m+n}}{\partial \epsilon^m \partial \bar{\epsilon}^n} \Lambda^A{}_B(\gamma, \bar{\gamma}; \epsilon, \bar{\epsilon}) \right] \\ &= \sum_{m,n=0}^{\infty} \frac{\epsilon^m \bar{\epsilon}^n}{m!n!} \left[\left(\dot{\gamma}^{c'} \hat{\nabla}_{c'} \right)^m \left(\dot{\bar{\gamma}}^d \hat{\nabla}_d \right)^n \hat{g}^A{}_{B'} \right]_{x' \to x}, \end{aligned} \tag{4.112} $$

where this coincidence limit is obtained by first taking the limit $\bar{\epsilon} \to 0$, followed by $\epsilon \to 0$. Assuming that our spacetime is sufficiently smooth, this reordering of limits is allowed. Keeping terms to only quadratic order yields [using the expressions for the coincidence limits of parallel propagators from equation (4.94)]

$$ {}_{\triangle}\hat{\Lambda}^A{}_B(\gamma, \bar{\gamma}; \epsilon, \bar{\epsilon}) = \delta^A{}_B - \frac{1}{2}\epsilon\bar{\epsilon}\dot{\gamma}^c\dot{\bar{\gamma}}^d \hat{R}^A{}_{Bcd} + O(\epsilon, \bar{\epsilon})^3. \tag{4.113} $$

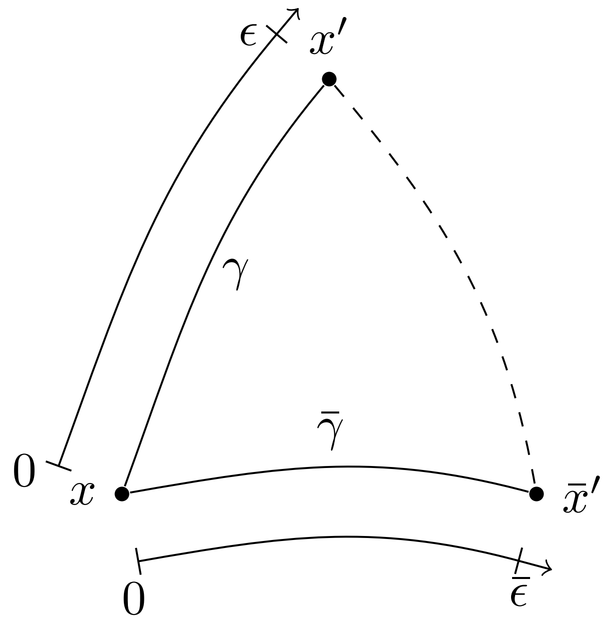

Figure 4.4: A nongeodesic triangle (generalizing figure 3 of [174]), where the two sides γ and $\bar{\gamma}$ are arbitrary curves (with affine parameter lengths ϵ and $\bar{\epsilon}$), and where the third side is formed by joining the two end points by the unique geodesic extending between them.

Note that equation (4.113) does not contain any acceleration terms at this order; moreover, it reduces to the results of [174] for the metric-compatible connection on the tangent bundle.

We next consider the holonomy around a square, such as that given in figure 4.5: this square is determined by two arbitrary curves γ and $\bar{\gamma}$, with the pairs of initial and final points, respectively, connected by the unique geodesics between them. The initial points are labeled $x = \gamma(0)$ and $\bar{x} = \bar{\gamma}(0)$, and the final points are labeled $x' = \gamma(\epsilon)$ and $\bar{x}' = \bar{\gamma}(\epsilon)$, and we assume that ϵ is small. We define two "separation vectors"

$$\xi^a(\bar{x}) \equiv -\sigma^a(\bar{x}) = g^a{}_{\bar{a}}\sigma^{\bar{a}}(x), \qquad \psi^a(\bar{x},\epsilon) \equiv -\sigma^a(\bar{x}'). \tag{4.114}$$

In terms of these quantities, the holonomy around this square is given by

$$\begin{aligned}{}_\square\hat{\Lambda}^A{}_B(\gamma,\bar{\gamma};\epsilon) &\equiv \hat{g}^A{}_{\bar{A}}\,{}_\triangle\left(\hat{\Lambda}^{-1}\right)^{\bar{A}}{}_{\bar{C}}(\bar{\gamma},\Gamma_{(\bar{x},x)};\epsilon,1)\hat{g}^{\bar{C}}{}_C\,{}_\triangle\hat{\Lambda}^C{}_B(\gamma,\Gamma_{(x,\bar{x}')};\epsilon,1) \\ &= \delta^A{}_B - \frac{\epsilon}{2}\left[\dot{\gamma}^c\psi^d(\bar{x},\epsilon)\hat{R}^A{}_{Bcd} + \hat{g}^A{}_{\bar{A}}\hat{g}^{\bar{B}}{}_B\dot{\bar{\gamma}}^{\bar{c}}g^{\bar{d}}{}_d\xi^d(\bar{x})\hat{R}^{\bar{A}}{}_{\bar{B}\bar{c}\bar{d}} + O(\boldsymbol{\xi},\boldsymbol{\psi})^2\right] + O(\epsilon^2).\end{aligned} \tag{4.115}$$

Note that we have traversed this loop in a way such that we can use equation (4.113), which was only established with two of the sides of the triangle being nongeodesic. First, note that, expanding

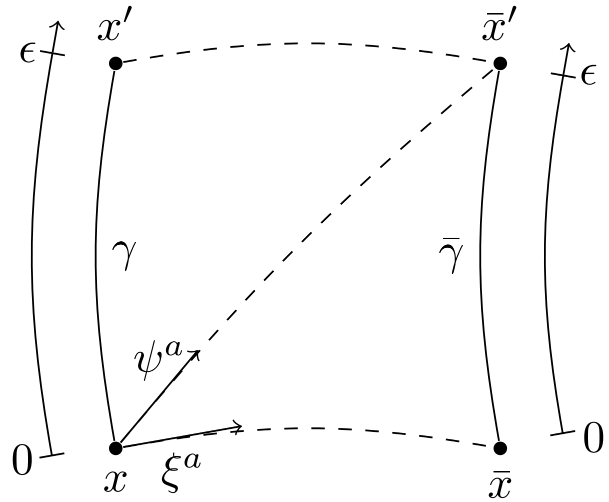

Figure 4.5: A nongeodesic square, with two sides γ and $\bar{\gamma}$ that are arbitrary curves (with equal affine parameter length ϵ), and where the other two sides are the unique geodesics between the two initial and final points, respectively. A third unique geodesic forms the diagonal. We denote the tangents (normalized such that the total affine parameter lengths are 1) to these unique geodesics at x by ξ^a and ψ^a.

the second term within square brackets in equation (4.115), one finds that

$$\hat{g}^A{}_{\bar{A}}\hat{g}^{\bar{B}}{}_B\hat{R}^{\bar{A}}{}_{\bar{B}\bar{c}\bar{d}} = g^c{}_{\bar{c}}g^d{}_{\bar{d}}\hat{R}^A{}_{Bcd} + O(\boldsymbol{\xi}). \tag{4.116}$$

Moreover, the expansion for $\psi^a(\bar{x},\epsilon)$ to lowest order in ϵ can be derived by noting that $\psi^a(\bar{x},0) = \xi^a(\bar{x})$:

$$\psi^a(\bar{x},\epsilon) = \xi^a(\bar{x}) + O(\epsilon). \tag{4.117}$$

Plugging these expressions into equation (4.115) gives

$$_\square\hat{\Lambda}^A{}_B(\gamma,\bar{\gamma};\epsilon) = \delta^A{}_B - \frac{\epsilon}{2}\left(\dot{\gamma}^c + g^c{}_{\bar{c}}\dot{\bar{\gamma}}^{\bar{c}}\right)\xi^d(\bar{x})\hat{R}^A{}_{Bcd} + O(\epsilon^2,\boldsymbol{\xi}^2). \tag{4.118}$$

Finally, consider the holonomy about the narrow loop in figure 4.6, which we will denote by $\hat{\Lambda}^A{}_B(\gamma,\bar{\gamma};\tau')$, as in section 4.1.3. The curves γ and $\bar{\gamma}$ are connected at the points $\gamma(\tau)$ and $\bar{\gamma}(\tau)$, as well as the points $\gamma(\tau')$ and $\bar{\gamma}(\tau')$, by the unique geodesics passing between these points. In terms of the holonomy for an infinitesimal square loop, and for any τ', and a given ϵ, we have that

$$\hat{\Lambda}^A{}_B(\gamma,\bar{\gamma};\tau'+\epsilon) = \hat{\Lambda}^A{}_C(\gamma,\bar{\gamma};\tau')\,{}_\gamma\hat{g}^C{}_{C'}\,{}_\square\hat{\Lambda}^{C'}{}_{D'}(\gamma,\bar{\gamma};\epsilon)\,{}_\gamma\hat{g}^{D'}{}_B. \tag{4.119}$$

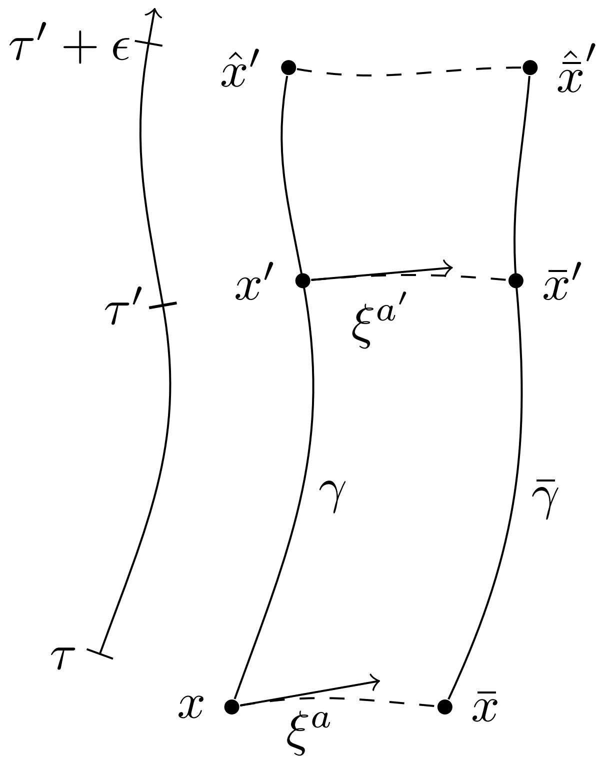

Figure 4.6: Two closely-separated worldlines γ and $\bar{\gamma}$, joined by unique geodesics between the start and end points of γ and $\bar{\gamma}$, respectively. The separation vector between the two worldlines is denoted by ξ^a.

Taking the limit $\epsilon \to 0$, we find the following differential equation for the holonomy of a narrow loop:

$$\frac{\mathrm{d}\hat{\Lambda}^A{}_B(\gamma,\bar{\gamma};\tau')}{\mathrm{d}\tau'} = \hat{\Lambda}^A{}_E(\gamma,\bar{\gamma};\tau')\,{}_\gamma\hat{g}^E{}_{E'}\,{}_\gamma\hat{g}^{B'}{}_B\hat{R}^{E'}{}_{B'c'd'}\xi^{c'}\dot{\gamma}^{d'} + O(\boldsymbol{\xi}',\dot{\boldsymbol{\xi}}')^2, \tag{4.120}$$

where we have used the fact that the separation vector $\xi^a(\bar{x})$ in equation (4.114) has been replaced by ξ^a, as well as equation (4.101). This differential equation can be solved iteratively. As in section 4.1.3.2, we consider the quantity

$$\hat{\Omega}^A{}_B(\gamma,\bar{\gamma};\tau') \equiv \hat{\Lambda}^A{}_B(\gamma,\bar{\gamma};\tau') - \delta^A{}_B \tag{4.121}$$

[as a generalization of equation (4.27)]. Keeping terms at first order in ξ^a and $\dot{\xi}^a$, we find that

$$\hat{\Omega}^A{}_B(\gamma,\bar{\gamma};\tau') = \int_\tau^{\tau'} \mathrm{d}\tau''\,{}_\gamma\hat{g}^A{}_{A''}\hat{R}^{A''}{}_{B''c''d''}\xi^{c''}\dot{\gamma}^{d''}\,{}_\gamma\hat{g}^{B''}{}_B + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2. \tag{4.122}$$

This confirms equation (4.28), and we find that

$$\hat{\Omega}^A{}_{Bc'd'}(\gamma) \equiv {}_\gamma\hat{g}^A{}_{A'}\hat{R}^{A'}{}_{B'c'd'}\,{}_\gamma\hat{g}^{B'}{}_B. \tag{4.123}$$

This discussion shows how the holonomy for narrow loops depends on the separation vector $\xi^{a''}$, for arbitrary $\tau'' \in [\tau, \tau']$. As one final note, we discuss how the holonomy depends on the initial separation ξ^a and relative velocity $\dot{\xi}^a$, assuming that γ and $\bar{\gamma}$ are both geodesic. First, we parametrize this dependence by defining the following expansion:

$$\hat{\Omega}^A{}_B(\gamma, \bar{\gamma}; \tau') \equiv \sum_{m=1}^{\infty}\sum_{k=0}^{m} {}_{\xi^k\dot{\xi}^{m-k}}\hat{\Omega}^A{}_{Bc_1\cdots c_k d_1\cdots d_{m-k}}(\gamma;\tau')\xi^{c_1}\cdots\xi^{c_k}\dot{\xi}^{d_1}\cdots\dot{\xi}^{d_{m-k}}. \tag{4.124}$$

As such, we find that [by equation (4.108)]

$${}_\xi\hat{\Omega}^A{}_{Bc}(\gamma;\tau') = \int_\tau^{\tau'} \mathrm{d}\tau''\hat{\Omega}^A{}_{Bc''d''}(\gamma)\,{}_\gamma K^{c''}{}_c\dot{\gamma}^{d''}. \tag{4.125}$$

It is apparent that ${}_{\dot{\xi}}\hat{\Omega}^A{}_{Bc}(\gamma;\tau')$ is given by the same expression as in equation (4.125), but with $(\tau''-\tau)\,{}_\gamma H^{c''}{}_c$ replacing ${}_\gamma K^{c''}{}_c$. However, there is another way to determine ${}_{\dot{\xi}}\hat{\Omega}^A{}_B(\gamma;\tau')$: using equation (4.89), one has that

$$\begin{aligned} {}_\xi\hat{\Omega}^A{}_{Bc}(\gamma;\tau') &= -\int_\tau^{\tau'} \mathrm{d}\tau''\hat{\Omega}^A{}_{Bc''d''}(\gamma)\dot{\gamma}^{d''}\frac{\mathrm{D}[(\tau''-\tau)\,{}_\gamma H^{c''}{}_c]}{\mathrm{d}\tau} \\ &= -\frac{\hat{\mathrm{D}}\,{}_{\dot{\xi}}\hat{\Omega}^A{}_{Bc}(\gamma;\tau')}{\mathrm{d}\tau}. \end{aligned} \tag{4.126}$$

Thus, we find that

$${}_{\dot{\xi}}\hat{\Omega}^A{}_{Bc}(\gamma;\tau') = \int_\tau^{\tau'} \mathrm{d}\tau''\,{}_\gamma\hat{g}^A{}_{A''}\,{}_\xi\hat{\Omega}^{A''}{}_{B''c''}(\gamma;\tau')\,{}_\gamma\hat{g}^{B''}{}_B\,{}_\gamma g^{c''}{}_c. \tag{4.127}$$

That is, given knowledge of ${}_\xi\hat{\Omega}^{A''}{}_{B''c''}(\gamma;\tau')$ and the parallel propagators for all $\tau'' \in [\tau,\tau']$, one can determine ${}_{\dot{\xi}}\hat{\Omega}^A{}_{Bc}(\gamma;\tau')$. Moreover, it contains one more time integral along γ than ${}_\xi\hat{\Omega}^A{}_{Bc}(\gamma;\tau')$, as expected based on our results in section 4.1.3.

4.2.2.3 Solution to the Mathisson-Papapetrou equations

In this section, we review the solution to the Mathisson-Papapetrou equations, to linear order in the spin, by adapting a proof from [148]. Throughout this derivation, for convenience, we consider the worldline of an arbitrary spinning particle, which we denote by Γ, and use unadorned indices at $\Gamma(\tau)$ (where τ is arbitrary). As such, unlike in equation (4.44), we do not place bars on any of the indices.

First, note that the Mathisson-Papapetrou equations [equation (4.44)] do not form a fully determined system of equations, as they contain 13 variables (four in p^a, six in j^{ab}, and three in $\dot{\Gamma}^a$), but only 10 equations. To solve for all of these variables (in particular $\dot{\Gamma}^a$), we would need three more equations, which are given by so-called *spin-supplementary conditions*. A commonly used spin supplementary condition is the *Tulczyjew condition* [168], which is given by enforcing

$$j^{ab}p_b = 0 \tag{4.128}$$

along the worldline, for all τ. This says that the mass dipole moment, measured in the rest frame determined by p^a, is zero. The intrinsic spin per unit mass is then defined with respect to this rest frame as well:

$$s^a \equiv -\frac{1}{2p^e p_e}\epsilon^{abcd}p_b j_{cd}. \tag{4.129}$$

We now seek to solve for $s^{a'}$ and $\ddot{\Gamma}^{a'}$, to leading order in the initial intrinsic spin per unit mass s^a. To begin, define the following notions of mass and mass ratio,

$$\mathcal{M}(\tau) \equiv -p^a\dot{\Gamma}_a, \qquad m(\tau) \equiv \sqrt{-p^a p_a}, \qquad \mu(\tau) \equiv \mathcal{M}(\tau)/m(\tau), \tag{4.130}$$

as well as a "dynamical" four-velocity

$$\mathcal{U}^a \equiv p^a/m(\tau). \tag{4.131}$$

Next, note that the definition of intrinsic spin per unit mass in equation (4.129) obeys the equation

$$s^a p_a = 0. \tag{4.132}$$

As such, by equation (4.128), s^a and p^a can be used to find j^{ab}:

$$\begin{aligned} \epsilon^{abcd}p_c s_d &= -\frac{1}{2}\epsilon^{dabc}\mathcal{U}_c\epsilon_{defg}\mathcal{U}^e j^{fg} = 3\mathcal{U}^{[a}j^{bc]}\mathcal{U}_c \\ &= -j^{ab}. \end{aligned} \tag{4.133}$$

We can therefore rewrite the Mathisson-Papapetrou equations (4.44) in terms of $\mathcal{U}^a$ and s^a:

$$\dot{\mathcal{U}}^a = -[\dot{m}(\tau)/m(\tau)]\mathcal{U}^a + (R^*)^a{}_{bcd}\dot{\Gamma}^b\mathcal{U}^c s^d, \tag{4.134a}$$

$$\dot{s}^a = -[\dot{m}(\tau)/m(\tau)]s^a + \mathcal{U}^a(R^*)_{bcde}s^b\dot{\Gamma}^c\mathcal{U}^d s^e, \tag{4.134b}$$

where we have used equation (4.133) and the orthogonality of $\mathcal{U}^a$ and $\dot{\mathcal{U}}^a$. On contracting the first equation with p_a, we obtain

$$\dot{m}(\tau) = -(R^*)_{abcd} p^a \dot{\Gamma}^b \mathcal{U}^c s^d, \tag{4.135}$$

so the second equation reads

$$\dot{s}^a = 2 s^{(a} (R^*)_{bcde} \mathcal{U}^{b)} \dot{\Gamma}^c \mathcal{U}^d s^e = O(\boldsymbol{s}^2). \tag{4.136}$$

We now solve equation (4.136) iteratively, assuming that the initial s^a is small. As such, we find that

$$s^{a'} = g^{a'}{}_a s^a. \tag{4.137}$$

This is the first quantity which we need.

For $\ddot{\Gamma}^a$, we start with taking a derivative of the Tulczyjew condition (4.128), using equation (4.44b):

$$0 = \dot{p}^a j_{ab} - [m(\tau)]^2 \dot{\Gamma}^a + \mathcal{M}(\tau) p^a, \tag{4.138}$$

which yields

$$\begin{aligned} \dot{\Gamma}^a &= \mu(\tau) \mathcal{U}^a + \epsilon^{abcd} \dot{p}_b \mathcal{U}_c s_d = \mu(\tau) \mathcal{U}^a + O(\boldsymbol{s}^2) \\ &= \mathcal{U}^a + O(\boldsymbol{s}^2), \end{aligned} \tag{4.139}$$

where in the second equality we have used equation (4.44a) and in the third equality we have used the fact that $\mu(\tau)$ is set by normalizing $\dot{\Gamma}^a \dot{\Gamma}_a = \mathcal{U}^a \mathcal{U}_a = -1$. From equation (4.135), we therefore have that

$$\dot{m}(\tau) = -m(\tau) (R^*)_{abcd} \mathcal{U}^a \mathcal{U}^b \mathcal{U}^c s^d + O(\boldsymbol{s}^2) = O(\boldsymbol{s}^2). \tag{4.140}$$

Thus, we find that

$$p^{a'} = m(\tau) \dot{\Gamma}^{a'} + O(\boldsymbol{s}^2). \tag{4.141}$$

This gives us another equation which we will find useful in section 4.2.3.3.

Finally, putting together equations (4.134a), (4.139), and (4.140), we find that the acceleration of the spinning particle is given by

$$\ddot{\Gamma}^{a'} = -(R^*)^{a'}{}_{c'b'd'} \dot{\Gamma}^{c'} \dot{\Gamma}^{d'} g^{b'}{}_b s^b + O(\boldsymbol{s}^2). \tag{4.142}$$

Although it is by no means necessary in this chapter, one could easily have performed all of these computations to $O(\boldsymbol{s})^2$ in order to get the next-to-leading-order behavior. At that order, the use of covariant bitensors, in particular the parallel propagator, becomes much more apparent.

4.2.3 Results

In this section, we provide explicit expressions for all of the persistent observables in section 4.1, which we give using the formalism of covariant bitensors reviewed in section 4.2.1, and based on the applications of covariant bitensors that we reviewed in section 4.2.2. These results hold in arbitrary spacetimes which transition from a flat region, to a curved region, and then another flat region. They are also necessarily perturbative in the separation, relative velocities, and accelerations of the observers. Later in this chapter, in section 4.3, we will give expressions that are valid in plane wave spacetimes, and there, we will be able to obtain *nonperturbative* results. Results that are valid, assuming weak curvature, were given in section 4.1. These results are particularly useful for discussing the feasibility of measuring these observables.

Before diving into these results, we make a brief note on notation. First, as persistent observables are defined with respect to an interval of proper time, we denote the initial time by τ_0 and the final time by τ_1; intermediate times are denoted by τ_2, τ_3, etc. For a curve γ, points $\gamma(\tau_n)$ are denoted by $x^{(n)}$, where $x^{(n)}$ is x with n primes, so $x \equiv \gamma(\tau_0)$, $x' \equiv \gamma(\tau_1)$, $x'' \equiv \gamma(\tau_2)$, etc. We also use a similar notation for worldlines with other diacritics; for example, $\bar{x}' \equiv \bar{\gamma}(\tau')$. When considering some arbitrary τ, we also use x and $\bar{x}$ for convenience.

4.2.3.1 Curve deviation

The first observable that we consider is the curve deviation observable of section 4.1.2. Most of the work has already been done in section 4.2.2.1, and so we simply put together the necessary equations and present the results.

We start with the definition of the curve deviation observable in equation (4.7). This definition depends on $\xi^{a'}$, and so we use the solution in equation (4.108). This solution depends on the nonlinear (and acceleration-dependent) source term that arises due to inserting the first-order solution (4.106) back into the nonlinear pieces of equation (4.105). Using the same notation as in equation (4.8), we find that

$$\Delta K^{a'}{}_b = {}_\gamma K^{a'}{}_b - {}_\gamma g^{a'}{}_b, \tag{4.143a}$$

$$\Delta H^{a'}{}_b = {}_\gamma H^{a'}{}_b - {}_\gamma g^{a'}{}_b, \tag{4.143b}$$

$$L^{a'}{}_{bc} = -\frac{1}{2}\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 (\tau_1 - \tau_2)\, {}_\gamma H^{a'}{}_{a''} \dot\gamma^{d''} \Big(S^{a''}{}_{b''c''d''e''} \dot\gamma^{e''}\, {}_\gamma K^{b''}{}_b\, {}_\gamma K^{c''}{}_c + 4 R^{a''}{}_{c''b''d''}\, {}_\gamma K^{b''}{}_{(b|} \frac{\mathrm{D}}{\mathrm{d}\tau_2}\, {}_\gamma K^{c''}{}_{|c)} \Big), \tag{4.143c}$$

$$N^{a'}{}_{bc} = -\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 (\tau_1 - \tau_2)\, {}_\gamma H^{a'}{}_{a''} \dot\gamma^{d''} \times \Bigg\{ (\tau_2 - \tau_0) S^{a''}{}_{b''c''d''e''} \dot\gamma^{e''}\, {}_\gamma K^{b''}{}_b\, {}_\gamma H^{c''}{}_c + 2 R^{a''}{}_{c''b''d''} \left({}_\gamma K^{b''}{}_b \frac{\mathrm{d}}{\mathrm{d}\tau_2} \left[(\tau_2 - \tau_0)\, {}_\gamma H^{c''}{}_c \right] + (\tau_2 - \tau_0)\, {}_\gamma H^{b''}{}_c \frac{\mathrm{d}}{\mathrm{d}\tau_2}\, {}_\gamma K^{c''}{}_b \right) \Bigg\}, \tag{4.143d}$$

$$M^{a'}{}_{bc} = -\frac{1}{2}\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 (\tau_1 - \tau_2)(\tau_2 - \tau_0)\, {}_\gamma H^{a'}{}_{a''} \dot\gamma^{d''} \Bigg\{ (\tau_2 - \tau_0) S^{a''}{}_{b''c''d''e''} \dot\gamma^{e''}\, {}_\gamma H^{b''}{}_b\, {}_\gamma H^{c''}{}_c + 4 R^{a''}{}_{c''b''d''}\, {}_\gamma H^{b''}{}_{(b|} \frac{\mathrm{d}}{\mathrm{d}\tau_2} \left[(\tau_2 - \tau_0)\, {}_\gamma H^{c''}{}_{|c)} \right] \Bigg\}, \tag{4.143e}$$

where $S^a{}_{bcde}$ was defined in equation (4.11). This procedure also confirms the second line of equation (4.8), where a very specific dependence on the accelerations of the two worldlines was presented. Finally, we note that equations (4.143a) and (4.143b) provide justification for the names "$\Delta K^{a'}{}_a$" and "$\Delta H^{a'}{}_a$".

4.2.3.2 Holonomies

We now consider the various holonomy observables that were defined in this chapter. For simplicity, we start with the holonomy $\Lambda^a{}_b(\gamma, \bar\gamma; \tau_1)$ of the metric-compatible connection. Most of the work has already been done in section 4.2.2.2 above, and we simply find that the relevant quantity is

$$\Omega^a{}_{bc'd'}(\gamma) = {}_\gamma g^a{}_{a'} R^{a'}{}_{b'c'd'}\, {}_\gamma g^b{}_{b'}. \tag{4.144}$$

Next, we consider the computation of the holonomy of transport of linear and angular momentum using equation (4.19). As in section 4.1.3, we denote by $\overset{\varkappa}{\nabla}_a$ the connection on the linear and angular momentum bundle for arbitrary $\varkappa$. Similarly, as was introduced in section 4.1.3.3, for $\varkappa = (0, 0, 0, 0)$, we use $\overset{0}{\nabla}_a$, and for $\varkappa = (1/2, 0, 0, 0)$, we use $\overset{1/2}{\nabla}_a$.

First, we calculate $\overset{\varkappa}{R}{}^A{}_{Bcd}$, starting from equation (4.95). However, we first note a minor simplification. Let

$$Z_{abcd} = V_{a[c}W_{d]b}, \tag{4.145}$$

where V_{ab} and W_{ab} are symmetric. Note that, in equation (4.20), $\overset{\varkappa}{K}{}^a{}_{bcd}$ is a sum of four terms, the first of which is just proportional to the Riemann tensor, and the latter three of which are of the form of Z_{abcd}. Now, it happens that Z_{abcd} can be written as two separate expressions, the first of which is the difference of two cyclic permutations of bcd, and the second of which is the difference of two cyclic permutations of acd:

$$Z_{abcd} = \frac{1}{2}(V_{ac}W_{db} - V_{ad}W_{bc}) = \frac{1}{2}(V_{ac}W_{db} - V_{da}W_{cb}). \tag{4.146}$$

As such, Z_{abcd} must vanish under cyclic permutations of bcd and acd, and so we find that

$$Z_{a[bcd]} = \frac{1}{3}(Z_{abcd} + Z_{acdb} + Z_{adbc}) = 0, \tag{4.147}$$

$$Z_{[a|b|cd]} = \frac{1}{3}(Z_{abcd} + Z_{cbda} + Z_{dbac}) = 0. \tag{4.148}$$

Thus, we find that $\overset{\varkappa}{K}_{abcd}$ satisfies the algebraic Bianchi identity:

$$\overset{\varkappa}{K}_{a[bcd]} = \overset{\varkappa}{K}_{[a|b|cd]} = 0 \tag{4.149}$$

(note that this is true even though $\overset{\varkappa}{K}_{abcd} \not\propto \overset{\varkappa}{K}_{bacd}$, since $\varkappa_2$ and $\varkappa_3$ are arbitrary). A simple consequence of equation (4.149) is that

$$2\overset{\varkappa}{K}_{a[cd]b} = \overset{\varkappa}{K}_{acdb} + \overset{\varkappa}{K}_{adbc} = -\overset{\varkappa}{K}_{abcd}, \tag{4.150}$$

and so we find

$$\overset{\varkappa}{R}{}^A{}_{Cef} = \begin{pmatrix} R^a{}_{cef} - 2\overset{\varkappa}{K}{}^a{}_{cef} & 2\nabla_{[e}\overset{\varkappa}{K}{}^a{}_{f]cd} \\ 0 & 2\delta^{[a}{}_{[c}R^{b]}{}_{d]ef} + 4\delta^{[a}{}_{[e}\overset{\varkappa}{K}{}^{b]}{}_{f]cd} \end{pmatrix}. \tag{4.151}$$

Given the parallel propagators with respect to $\breve{\breve{\nabla}}_a$, the values of $\breve{\breve{\Omega}}^A{}_{Bc'd'}(\gamma)$ are therefore given by equations (4.123) and (4.151):

$$\begin{aligned}
\underset{PP}{\breve{\breve{\Omega}}}{}^a{}_{ce'f'}(\gamma) = {}& {}_\gamma\underset{PP}{\breve{\breve{g}}}{}^a{}_{a'} \left[\left(R^{a'}{}_{c'e'f'} - 2\breve{\breve{K}}{}^{a'}{}_{c'e'f'} \right) {}_\gamma\underset{PP}{\breve{\breve{g}}}{}^{c'}{}_c + 2\nabla_{e'}\breve{\breve{K}}{}^{a'}{}_{f'c'd'}\,{}_\gamma\underset{JP}{\breve{\breve{g}}}{}^{c'd'}{}_c \right] \\
& + 2\,{}_\gamma\underset{PJ}{\breve{\breve{g}}}{}^a{}_{a'b'} \left[\delta^{a'}{}_{c'} R^{b'}{}_{d'e'f'} + 2\delta^{a'}{}_{e'} \breve{\breve{K}}{}^{b'}{}_{f'c'd'} \right] {}_\gamma\underset{JP}{\breve{\breve{g}}}{}^{c'd'}{}_c, \qquad (4.152a) \\
\underset{PJ}{\breve{\breve{\Omega}}}{}^a{}_{cde'f'}(\gamma) = {}& {}_\gamma\underset{PP}{\breve{\breve{g}}}{}^a{}_{a'} \left[\left(R^{a'}{}_{c'e'f'} - 2\breve{\breve{K}}{}^{a'}{}_{c'e'f'} \right) {}_\gamma\underset{PJ}{\breve{\breve{g}}}{}^{c'}{}_{cd} + 2\nabla_{e'}\breve{\breve{K}}{}^{a'}{}_{f'c'd'}\,{}_\gamma\underset{JJ}{\breve{\breve{g}}}{}^{c'd'}{}_{cd} \right] \\
& + 2\,{}_\gamma\underset{PJ}{\breve{\breve{g}}}{}^a{}_{a'b'} \left[\delta^{a'}{}_{c'} R^{b'}{}_{d'e'f'} + 2\delta^{a'}{}_{e'} \breve{\breve{K}}{}^{b'}{}_{f'c'd'} \right] {}_\gamma\underset{JJ}{\breve{\breve{g}}}{}^{c'd'}{}_{cd}, \qquad (4.152b) \\
\underset{JP}{\breve{\breve{\Omega}}}{}^{ab}{}_{ce'f'}(\gamma) = {}& {}_\gamma\underset{JP}{\breve{\breve{g}}}{}^{ab}{}_{a'} \left[\left(R^{a'}{}_{c'e'f'} - 2\breve{\breve{K}}{}^{a'}{}_{c'e'f'} \right) {}_\gamma\underset{PP}{\breve{\breve{g}}}{}^{c'}{}_c + 2\nabla_{e'}\breve{\breve{K}}{}^{a'}{}_{f'c'd'}\,{}_\gamma\underset{JP}{\breve{\breve{g}}}{}^{c'd'}{}_c \right] \\
& + 2\,{}_\gamma\underset{JJ}{\breve{\breve{g}}}{}^{ab}{}_{a'b'} \left[\delta^{a'}{}_{c'} R^{b'}{}_{d'e'f'} + 2\delta^{a'}{}_{e'} \breve{\breve{K}}{}^{b'}{}_{f'c'd'} \right] {}_\gamma\underset{JP}{\breve{\breve{g}}}{}^{c'd'}{}_c, \qquad (4.152c) \\
\underset{JJ}{\breve{\breve{\Omega}}}{}^{ab}{}_{cde'f'}(\gamma) = {}& {}_\gamma\underset{JP}{\breve{\breve{g}}}{}^{ab}{}_{a'} \left[\left(R^{a'}{}_{c'e'f'} - 2\breve{\breve{K}}{}^{a'}{}_{c'e'f'} \right) {}_\gamma\underset{PJ}{\breve{\breve{g}}}{}^{c'}{}_{cd} + 2\nabla_{e'}\breve{\breve{K}}{}^{a'}{}_{f'c'd'}\,{}_\gamma\underset{JJ}{\breve{\breve{g}}}{}^{c'd'}{}_{cd} \right] \\
& + 2\,{}_\gamma\underset{JJ}{\breve{\breve{g}}}{}^{ab}{}_{a'b'} \left[\delta^{a'}{}_{c'} R^{b'}{}_{d'e'f'} + 2\delta^{a'}{}_{e'} \breve{\breve{K}}{}^{b'}{}_{f'c'd'} \right] {}_\gamma\underset{JJ}{\breve{\breve{g}}}{}^{c'd'}{}_{cd}. \qquad (4.152d)
\end{aligned}$$

In most cases, we cannot analytically solve for these parallel propagators nonperturbatively in the Riemann tensor. The results presented in section 4.1.3 are perturbative, assuming the curvature is weak along the worldline. In such a case, solutions to equation (4.63) can be truncated at a low order in the Riemann tensor, and one has that

$$ {}_\gamma\breve{\breve{g}}{}^{A'}{}_A = {}_\gamma\overset{0}{g}{}^{A'}{}_A + O(\boldsymbol{R}). \qquad (4.153)$$

Since ${}_\gamma\overset{0}{A}{}^A{}_{B'}$ is nilpotent, one can show that

$$\begin{aligned}
{}_\gamma\overset{0}{g}{}^{A'}{}_A &= \begin{pmatrix} {}_\gamma g^{a'}{}_a & 0 \\ -2\displaystyle\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2\, {}_\gamma g^{[a'}{}_{a''}\,{}_\gamma g^{b']}{}_a \dot{\gamma}^{a''} & {}_\gamma g^{[a'}{}_a\,{}_\gamma g^{b']}{}_b \end{pmatrix}, \qquad (4.154a) \\
{}_\gamma\overset{0}{g}{}^A{}_{A'} &= \begin{pmatrix} {}_\gamma g^a{}_{a'} & 0 \\ 2\displaystyle\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2\, {}_\gamma g^{[a}{}_{a''}\,{}_\gamma g^{b]}{}_{a'} \dot{\gamma}^{a''} & {}_\gamma g^{[a}{}_{a'}\,{}_\gamma g^{b]}{}_{b'} \end{pmatrix}. \qquad (4.154b)
\end{aligned}$$

This gives the expressions in equations (4.29) and (4.30).

We now specialize to the case of affine transport. Equation (4.152) simplifies to

$$\overset{0}{\Omega}{}^{A}{}_{Ce'f'}(\gamma) = \begin{pmatrix} \Omega^{a}{}_{bc'd'}(\gamma) & 0 \\ 2\delta^{[a}{}_{c}\,{}_{\gamma}g^{b]}{}_{b'}R^{b'}{}_{g'e'f'}\displaystyle\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2\,{}_{\gamma}g^{g'}{}_{g''}\dot{\gamma}^{g''} & 2\delta^{[a}{}_{[c}\Omega^{b]}{}_{d]e'f'}(\gamma) \end{pmatrix}, \tag{4.155}$$

which yields, by equations (4.28) and (4.154), followed by an integration by parts,

$$\overset{0}{\Lambda}{}^{A}{}_{C}(\gamma,\bar{\gamma};\tau_1) = \begin{pmatrix} \Lambda^{a}{}_{c}(\gamma,\bar{\gamma};\tau_1) & 0 \\ 2\displaystyle\int_{\tau_0}^{\tau_1}\mathrm{d}\tau_2\int_{\tau_2}^{\tau_1}\mathrm{d}\tau_3\delta^{[a}{}_{c}\,{}_{\gamma}g^{b]}{}_{b'''}R^{b'''}{}_{g'''e'''f'''}\,{}_{\gamma}g^{g'''}{}_{g''}\dot{\gamma}^{g''}\dot{\gamma}^{e'''}\xi^{f'''} & 2\delta^{[a}{}_{[c}\Lambda^{b]}{}_{d]}(\gamma,\bar{\gamma};\tau_1) \end{pmatrix} + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2. \tag{4.156}$$

At this point, let us assume that γ is geodesic; in this case, one has that

$${}_{\gamma}g^{a'''}{}_{a''}\dot{\gamma}^{a''} = \dot{\gamma}^{a'''}, \tag{4.157}$$

and so [using equation (4.105)] one finds that

$$\underset{PP}{\overset{0}{\Lambda}}{}^{a}{}_{c}(\gamma,\bar{\gamma};\tau_1) = \Lambda^{a}{}_{c}(\gamma,\bar{\gamma};\tau_1) + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2, \tag{4.158a}$$

$$\underset{JP}{\overset{0}{\Lambda}}{}^{ab}{}_{c}(\gamma,\bar{\gamma};\tau_1) = 2\delta^{[a}{}_{c}\delta^{b]}{}_{e}\left\{{}_{\gamma}g^{e}{}_{e'}\left[(\tau_1-\tau_0)\dot{\xi}^{e'} - \xi^{e'}\right] + \xi^{e}\right\} + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2, \tag{4.158b}$$

$$\underset{JJ}{\overset{0}{\Lambda}}{}^{ab}{}_{cd}(\gamma,\bar{\gamma};\tau_1) = 2\delta^{[a}{}_{[c}\Lambda^{b]}{}_{d]}(\gamma,\bar{\gamma};\tau_1) + O(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})^2. \tag{4.158c}$$

Note that the expression for $\underset{JP}{\overset{0}{\Lambda}}{}^{ab}{}_{c}(\gamma,\bar{\gamma};\tau_1)$ is related to the displacement memory and relative velocity observables, as it is written in terms of $\xi^{a'}$ and $\dot{\xi}^{a'}$. There are additional complications in this expression in the presence of acceleration. Both $\underset{PP}{\overset{0}{\Lambda}}{}^{a}{}_{c}(\gamma,\bar{\gamma};\tau_1)$ and $\underset{JJ}{\overset{0}{\Lambda}}{}^{ab}{}_{cd}(\gamma,\bar{\gamma};\tau_1)$ depend upon just the usual holonomy, and therefore they contain the same information as the Lorentz transformation observable.

The holonomy for dual Killing transport similarly has a solution that is nonperturbative in the Riemann tensor, since the parallel propagators with respect to this connection are related to the Jacobi propagators, assuming that γ is geodesic. This is shown as follows: suppose that we have some ξ_a and ω_{ab} defined as tensor fields along γ such that

$$Y_A \equiv \begin{pmatrix} \xi_a & \tfrac{1}{2}\omega_{ab} \end{pmatrix}, \qquad \dot{\gamma}^b\overset{1/2}{\nabla}_b Y_A = 0. \tag{4.159}$$

Note that this is *very* reminiscent of the discussion of the Killing transport equations in section 4.1.3.3. This implies ξ_a and ω_{ab} satisfy

$$\dot{\gamma}^b \nabla_b \xi_a = \dot{\gamma}^b \omega_{ba}, \tag{4.160a}$$

$$\dot{\gamma}^c \nabla_c \omega_{ab} = R^d{}_{cab} \xi_d \dot{\gamma}^c. \tag{4.160b}$$

Then we have that (as γ is geodesic)

$$\frac{\mathrm{D}^2 \xi_a}{\mathrm{d}\tau^2} = -R^b{}_{cad} \dot{\gamma}^c \dot{\gamma}^d \xi_b. \tag{4.161}$$

Note that by raising a (which commutes with $\mathrm{D}/\mathrm{D}\tau$), we obtain equation (4.67). The Jacobi propagators therefore give the solution to equation (4.161):

$$\xi_{a'} = {}_{\gamma}K_{a'}{}^{a} \xi_a + (\tau_1 - \tau_0) \dot{\gamma}^a \, {}_{\gamma}H_{a'}{}^{b} \omega_{ab} \tag{4.162}$$

(this follows from the fact that $g^a{}_{a'} = g_{a'b'} g^{ab} g^{b'}{}_b$, a consequence of the metric-compatibility of ∇_a). Integrating equation (4.160b), we find that

$$\begin{aligned} \omega_{a'b'} &= {}_{\gamma}g^a{}_{a'} \, {}_{\gamma}g^b{}_{b'} \omega_{ab} + \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \, {}_{\gamma}g^{a''}{}_{a'} \, {}_{\gamma}g^{b''}{}_{b'} R^{c''}{}_{d''a''b''} \dot{\gamma}^{d''} \xi_{c''} \\ &= {}_{\gamma}g^a{}_{a'} \, {}_{\gamma}g^b{}_{b'} \omega_{ab} + \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \Omega_{a'b'}{}^{c''d''}(\gamma) \xi_{c''} \dot{\gamma}_{d''}. \end{aligned} \tag{4.163}$$

Equations (4.162) and (4.163) give $\xi_{a'}$ and $\omega_{a'b'}$ as linear functions of ξ_a and ω_{ab}, and we can use them to write the parallel propagator as follows:

$$ {}_{\gamma}\overset{1/2}{g}{}^{A}{}_{A'} = \begin{pmatrix} {}_{\gamma}K_{a'}{}^{a} & \frac{1}{2} \, {}_{\xi}\Omega_{cd}{}^{a}(\gamma;\tau_1) \, {}_{\gamma}g^c{}_{a'} \, {}_{\gamma}g^d{}_{b'} \\ 2(\tau_1 - \tau_0) \dot{\gamma}^{[a} \, {}_{\gamma}H_{a'}{}^{b]} & {}_{\gamma}g^{[a}{}_{a'} \, {}_{\gamma}g^{b]}{}_{b'} + \dot{\gamma}^{[a} \, {}_{\dot{\xi}}\Omega_{cd}{}^{b]}(\gamma;\tau_1) \, {}_{\gamma}g^c{}_{a'} \, {}_{\gamma}g^d{}_{b'} \end{pmatrix}. \tag{4.164}$$

It is possible to invert this matrix, but a simpler approach is to switch τ_0 with τ_1, which yields

$$ {}_{\gamma}\overset{1/2}{g}{}^{A'}{}_{A} = \begin{pmatrix} {}_{\gamma}K_{a}{}^{a'} & \frac{1}{2} \, {}_{\xi}\Omega_{c'd'}{}^{a'}(\gamma;\tau_0) \, {}_{\gamma}g^{c'}{}_{a} \, {}_{\gamma}g^{d'}{}_{b} \\ -2(\tau_1 - \tau_0) \dot{\gamma}^{[a'} \, {}_{\gamma}H_{a}{}^{b']} & {}_{\gamma}g^{[a'}{}_{a} \, {}_{\gamma}g^{b']}{}_{b} + \dot{\gamma}^{[a'} \, {}_{\dot{\xi}}\Omega_{c'd'}{}^{b']}(\gamma;\tau_0) \, {}_{\gamma}g^{c'}{}_{a} \, {}_{\gamma}g^{d'}{}_{b} \end{pmatrix}. \tag{4.165}$$

Note that, to zeroth order in the Riemann tensor, these two equations agree with equation (4.154). Moreover, these expressions appear to be connected to the holonomy with respect to the metric-compatible connection, through the functions ${}_{\xi}\Omega^a{}_{bc}(\gamma;\tau')$ and ${}_{\dot{\xi}}\Omega^a{}_{bc}(\gamma;\tau')$ that appear. This fact will be crucial in section 4.3.3.2, where we use this to determine the dual Killing transport holonomy in plane wave spacetimes.

To complete the calculation of the holonomy for dual Killing transport, we further simplify our expression for $\overset{1/2}{R}{}^{A}{}_{Bcd}$. Note that $\underset{PP}{\overset{1/2}{R}}{}^{a}{}_{cef} = 0$, and

$$2\nabla_{[e}R_{|a|f]cd} = \nabla_e R_{afcd} + \nabla_f R_{eacd} = \nabla_a R_{efcd}, \tag{4.166}$$

by the differential Bianchi identity, so $\underset{PJ}{\overset{1/2}{R}}{}^{a}{}_{cdef} = \frac{1}{2}\nabla^a R_{efcd} = \frac{1}{2}\nabla^a R_{cdef}$. Using the same notation as in equation (4.28), we obtain our final result in terms of the parallel and Jacobi propagators and the curvature along the worldline. First, we define

$$\begin{aligned}
{}_{\gamma}A^{a}{}_{e'f'c'd'} &= \frac{1}{4}\,{}_{\gamma}K_{a'}{}^{a}\nabla^{a'}R_{e'f'c'd'} - {}_{\xi}\Omega_{cd}{}^{a}(\gamma;\tau_1)\,{}_{\gamma}g^{c}{}_{b'}\,{}_{\gamma}g^{d}{}_{[c'}R^{b'}{}_{d']e'f'} + ([c'd'] \leftrightarrow [e'f']), &(4.167a)\\
{}_{\gamma}B^{ab}{}_{e'f'c'd'} &= \frac{1}{2}(\tau_1-\tau_0)\dot\gamma^{[a}\,{}_{\gamma}H_{a'}{}^{b]}\nabla^{a'}R_{e'f'c'd'} - 2\left[\delta^{[a}{}_{c}\delta^{b]}{}_{d} + \dot\gamma^{[a}\,{}_{\dot\xi}\Omega_{cd}{}^{b]}(\gamma;\tau_1)\right]{}_{\gamma}g^{c}{}_{b'}\,{}_{\gamma}g^{d}{}_{[c'}R^{b'}{}_{d']e'f'}\\
&\quad + ([c'd'] \leftrightarrow [e'f']), &(4.167b)
\end{aligned}$$

where "$+([c'd'] \leftrightarrow [e'f'])$" means "add all the previous terms in the sum, but with $[c'd']$ and $[e'f']$ switched." In terms of these two bitensors, we have that

$$\begin{aligned}
\underset{PP}{\overset{1/2}{\Omega}}{}^{a}{}_{ce'f'}(\gamma) &= -2(\tau_1-\tau_0)\,{}_{\gamma}A^{a}{}_{e'f'c'd'}\dot\gamma^{c'}\,{}_{\gamma}H_{c}{}^{d'}, &(4.168a)\\
\underset{PJ}{\overset{1/2}{\Omega}}{}^{a}{}_{cde'f'}(\gamma) &= {}_{\gamma}A^{a}{}_{e'f'c'd'}\left[\delta^{c'}{}_{g'}\delta^{d'}{}_{h'} + \dot\gamma^{[c'}\,{}_{\dot\xi}\Omega_{g'h'}{}^{d']}(\gamma;\tau_0)\right]{}_{\gamma}g^{g'}{}_{c}\,{}_{\gamma}g^{h'}{}_{d}, &(4.168b)\\
\underset{JP}{\overset{1/2}{\Omega}}{}^{ab}{}_{ce'f'}(\gamma) &= -2(\tau_1-\tau_0)\,{}_{\gamma}B^{ab}{}_{c'd'e'f'}\dot\gamma^{c'}\,{}_{\gamma}H_{c}{}^{d'}, &(4.168c)\\
\underset{JJ}{\overset{1/2}{\Omega}}{}^{ab}{}_{cde'f'}(\gamma) &= {}_{\gamma}B^{ab}{}_{c'd'e'f'}\left[\delta^{c'}{}_{g'}\delta^{d'}{}_{h'} + \dot\gamma^{[c'}\,{}_{\dot\xi}\Omega_{g'h'}{}^{d']}(\gamma;\tau_0)\right]{}_{\gamma}g^{g'}{}_{c}\,{}_{\gamma}g^{h'}{}_{d}. &(4.168d)
\end{aligned}$$

4.2.3.3 Spinning particles

We now consider the procedure outlined in section 4.1.4: an observer measures the separation ξ^a from an initially comoving spinning test particle, as well as its linear momentum p^a and spin per unit mass s^a. At some later point in time, the observer performs these measurements again. The persistent observables in this case are the differences between the initial and final measurements.

Given the results of section 4.2.2.3, along with sections 4.2.2.1 and 4.2.2.2, we can compute the observables in a relatively straightforward manner. First, we note that the initially measured momentum p^a and intrinsic spin s^a are given by

$$p^a = m g^a{}_{\bar{a}} \dot{\bar{\gamma}}^{\bar{a}} = m \dot{\gamma}^a, \qquad s^a = g^a{}_{\bar{a}} s^{\bar{a}}, \tag{4.169}$$

where the first equation assumes that the observer and spinning particle are initially comoving. The initial mass m is defined by $m^2 \equiv -\bar{p}^a \bar{p}_a$. Similarly,

$$p^{a'} = m g^a{}_{\bar{a}} \dot{\bar{\gamma}}^{\bar{a}'} + O(\boldsymbol{s}^2), \qquad s^{a'} = g^{a'}{}_{\bar{a}'} g^{\bar{a}'}{}_{\bar{a}} s^{\bar{a}} + O(\boldsymbol{s}^2), \tag{4.170}$$

where we have used the fact that, ignoring terms of $O(\boldsymbol{s}^2)$, m is constant and $s^{\bar{a}'}$ is parallel-transported.

The differences between the initial and final measurements are likewise defined by parallel transport:

$$\Delta \xi^a_{\mathrm{S}} \equiv \xi^{a'} - {}_\gamma g^{a'}{}_a \xi^a, \tag{4.171a}$$

$$\Delta p^a \equiv p^{a'} - {}_\gamma g^{a'}{}_a p^a, \tag{4.171b}$$

$$\Delta s^a \equiv s^{a'} - {}_\gamma g^{a'}{}_a s^a. \tag{4.171c}$$

For the intrinsic spin, this last equation implies that

$$\begin{aligned}
\Delta s^{a'} &= (g^{a'}{}_{\bar{a}'}\, {}_{\bar{\gamma}} g^{\bar{a}'}{}_{\bar{a}} g^{\bar{a}}{}_a - {}_\gamma g^{a'}{}_a) s^a + O(\boldsymbol{s})^2 \\
&= {}_\gamma g^{a'}{}_a \left[\left(\Lambda^{-1}\right)^a{}_b(\gamma, \bar{\gamma}; \tau_1) - \delta^a{}_b \right] s^b + O(\boldsymbol{s})^2 \\
&= -\,{}_\gamma g^{a'}{}_a \left[\Omega^a{}_b(\gamma, \bar{\gamma}; \tau_1) + O(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})^2 \right] s^b + O(\boldsymbol{s})^2.
\end{aligned} \tag{4.172}$$

This proves equation (4.46c), and shows that $\Delta s^{a'}$, to leading order in spin, contains the same information as the holonomy with respect to the metric-compatible connection. Furthermore, for the momentum, equation (4.101) implies that

$$\begin{aligned}
\Delta p^{a'} &= m \left(\dot{\xi}^{a'} + \dot{\gamma}^{a'} \right) - {}_\gamma g^{a'}{}_a p^a + O(\boldsymbol{s}^2) \\
&= m \dot{\xi}^{a'} + O(\boldsymbol{s}^2) = m \frac{\mathrm{D} \Delta \xi^{a'}_{\mathrm{S}}}{\mathrm{d}\tau_1} + O(\boldsymbol{s}^2),
\end{aligned} \tag{4.173}$$

which proves equation (4.46b); therefore, the computation of $\Delta p^{a'}$ is trivial once $\Delta \xi^{a'}_{\mathrm{S}}$ is known.

The separation evolves using equation (4.105). To compute $\Delta\xi_{\rm S}^{a'}$, we first need to calculate the acceleration of the spinning test particle to the relevant order:

$$\begin{aligned} g^{a'}{}_{\bar a'}\ddot{\bar\gamma}^{\bar a'} &= -g^{a'}{}_{\bar a'}(R^*)^{\bar a'}{}_{\bar c'\bar b'\bar d'}\dot{\bar\gamma}^{\bar c'}\dot{\bar\gamma}^{\bar d'}{}_{\bar\gamma}g^{\bar b'}{}_{\bar b}g^{\bar b}{}_b s^b + O(\boldsymbol{s})^2 \\ &= -\left[(R^*)^{a'}{}_{c'b'd'} + \xi^{e'}\nabla_{e'}(R^*)^{a'}{}_{c'b'd'}\right]\left[\dot\gamma^{c'}\dot\gamma^{d'} + 2\dot\gamma^{(c'}\dot\xi^{d')}\right]{}_\gamma g^{b'}{}_e(\Lambda^{-1})^e{}_b(\gamma,\bar\gamma;\tau_1)s^b \\ &\quad + O(\boldsymbol{s},\boldsymbol{\xi}^2)^2. \end{aligned} \tag{4.174}$$

To derive equation (4.174), we have used the definition of the holonomy, equation (4.101), and an expansion of the Riemann tensor off of γ. Now, we use equation (4.122) and the solution to the geodesic equation in equation (4.108) to write equation (4.174) in terms of ξ^a and s^a:

$$\begin{aligned} g^{a'}{}_{\bar a'}\ddot{\bar\gamma}^{\bar a'} = -\Bigg\{&(R^*)^{a'}{}_{c'b'd'}\,{}_\gamma g^{b'}{}_b\dot\gamma^{c'} \\ &+\Bigg[{}_\gamma K^{c'}{}_c\nabla_{c'}(R^*)^{a'}{}_{d'b'e'}\dot\gamma^{e'}\,{}_\gamma g^{b'}{}_b \\ &\qquad - (R^*)^{a'}{}_{(c'|b'|d')}\int_{\tau_0}^{\tau_1}\mathrm{d}\tau_2 R^{c''}{}_{b''f''g''}\,{}_\gamma G_b{}^{b'c'b''}{}_{c''}\dot\gamma^{g''}\,{}_\gamma K^{f''}{}_c\Bigg]\xi^c + O(\boldsymbol{\xi}^2)\Bigg\}s^b\dot\gamma^{d'} \\ &+ O(\boldsymbol{s})^2, \end{aligned} \tag{4.175}$$

where

$${}_\gamma G_a{}^{b'c'd''}{}_{e''} \equiv {}_\gamma g^{b'}{}_{e''}\dot\gamma^{c'}\,{}_\gamma g^{d''}{}_a + 2\dot\gamma^{d''}\,{}_\gamma g^{c'}{}_{e''}\,{}_\gamma g^{b'}{}_a. \tag{4.176}$$

Using equation (4.108), we find that our observables [using the notation in equation (4.46)] are given by

$$\Upsilon^{a'}{}_b = -\int_{\tau_0}^{\tau_1}\mathrm{d}\tau_2(\tau_1-\tau_2)\,{}_\gamma H^{a'}{}_{a''}(R^*)^{a''}{}_{c''b''d''}\dot\gamma^{c''}\dot\gamma^{d''}\,{}_\gamma g^{b''}{}_b, \tag{4.177a}$$

$$\begin{aligned} \Psi^{a'}{}_{bc} = -\int_{\tau_0}^{\tau_1}\mathrm{d}\tau_2(\tau_1-\tau_2)\,{}_\gamma H^{a'}{}_{a''}\Bigg[&{}_\gamma K^{c''}{}_c\nabla_{c''}(R^*)^{a''}{}_{d''b''e''}\dot\gamma^{e''}\,{}_\gamma g^{b''}{}_b \\ &- (R^*)^{a''}{}_{(c''|b''|d'')} \\ &\times\int_{\tau_0}^{\tau_2}\mathrm{d}\tau_3 R^{c'''}{}_{b'''f'''g'''}\,{}_\gamma G_b{}^{b''c''b'''}{}_{c'''}\dot\gamma^{g'''}\,{}_\gamma K^{f'''}{}_c\Bigg]\dot\gamma^{d''}. \end{aligned} \tag{4.177b}$$

As these results are given in terms of the Riemann tensor and the fundamental bitensors (parallel and Jacobi propagators), they can be computed with relative ease in spacetimes in which these bitensors are known.

4.2.3.4 Modifications Due to Acceleration

Many results in the previous sections, except for our spinning test particle observable, we been specialized to the case where all curves used to define the observables are geodesic. We now consider the effects of acceleration by determining how to write observables involving acceleration in terms of observables with no acceleration. To do so, note that, given a curve γ and a proper time τ_0, there is a unique geodesic $\hat{\gamma}$ that intersects γ at τ_0 that has the same four-velocity at that point. If an observable is defined with respect to two accelerated curves γ and $\bar{\gamma}$, we show that this observable can be written in terms of $\hat{\gamma}$ and $\hat{\bar{\gamma}}$.

This process is most easily done for the holonomy observable, which we consider for an arbitrary connection $\widetilde{\nabla}_a$ on some arbitrary vector bundle, as was done in section 4.2.2.2 (we do not denote this connection with a hat, as was done in section 4.2.2.2, as we are using hats to denote something else in this section). Using the fact that the initial and final regions are flat, we find that

$$\widetilde{\Lambda}^A{}_B(\gamma;\bar{\gamma};\tau_1) = \widetilde{g}^A{}_{\bar{A}}\widetilde{\Lambda}^{\bar{A}}{}_{\bar{C}}(\hat{\bar{\gamma}},\bar{\gamma};\tau_1)\widetilde{g}^{\bar{C}}{}_C\widetilde{\Lambda}^C{}_D(\hat{\gamma},\hat{\bar{\gamma}};\tau_1)\left(\widetilde{\Lambda}^{-1}\right)^D{}_B(\hat{\gamma},\gamma;\tau_1). \tag{4.178}$$

The only pieces that occur here that nontrivially modify the holonomy $\widetilde{\Lambda}^A{}_B(\hat{\gamma},\hat{\bar{\gamma}};\tau_1)$ are given by

$$\left(\widetilde{\Lambda}^{\pm 1}\right)^A{}_B(\hat{\gamma},\gamma;\tau_1) = \delta^A{}_B \pm {}_{\ddot{\gamma}}\widetilde{\Omega}^A{}_B(\hat{\gamma},\gamma;\tau_1) + O(\ddot{\gamma}^2), \tag{4.179}$$

where

$$\begin{aligned} {}_{\ddot{\gamma}}\widetilde{\Omega}^A{}_B(\hat{\gamma},\gamma;\tau_1) = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \Big\{ &(\tau_0-\tau_2)\, {}_{\hat{\gamma}}H^c{}_{\hat{c}''}\left[{}_{\xi}\widetilde{\Omega}^A{}_{Bc}(\hat{\gamma};\tau_1) - {}_{\xi}\widetilde{\Omega}^A{}_{Bc}(\hat{\gamma};\tau_2)\right] \\ &+ \frac{\mathrm{D}}{\mathrm{d}\tau}\left[(\tau_0-\tau_2)\, {}_{\hat{\gamma}}H^c{}_{\hat{c}''}\right]\left[{}_{\dot{\xi}}\widetilde{\Omega}^A{}_{Bc}(\hat{\gamma};\tau_1) - {}_{\dot{\xi}}\widetilde{\Omega}^A{}_{Bc}(\hat{\gamma};\tau_2)\right]\Big\} g^{\hat{c}''}{}_{c''}\ddot{\gamma}^{c''}. \end{aligned} \tag{4.180}$$

This result can shown using techniques very similar to those that derive equation (4.127), along with the dependence of the separation on acceleration resulting from equation (4.108).

We now consider the curve deviation observable, starting with the various pieces that go into the definition of the curve deviation in equation (4.7). Since it is crucial in this case, we make the dependence on the curves of the separation and the curve deviation observable explicit. First, one can show that, since the final region is flat,

$$\xi^{a'}(\gamma,\bar{\gamma}) = g^{a'}{}_{\hat{a}'}\left[\xi^{\hat{a}'}(\hat{\gamma},\hat{\bar{\gamma}}) + g^{\hat{a}'}{}_{\hat{\bar{a}}'}\xi^{\hat{\bar{a}}'}(\hat{\bar{\gamma}},\bar{\gamma}) - \xi^{\hat{a}'}(\hat{\gamma},\gamma)\right]. \tag{4.181}$$

Using the holonomy, we also find

$$ {}_\gamma g^{a'}{}_a = g^{a'}{}_{\hat{a}'}\, {}_{\hat{\gamma}} g^{\hat{a}'}{}_b \left(\Lambda^{-1}\right)^b{}_a(\hat{\gamma}, \gamma; \tau_1), \tag{4.182} $$

and moreover

$$ {}_\gamma g^{a'}{}_{a'''} = g^{a'}{}_{\hat{a}'}\, {}_{\hat{\gamma}} g^{\hat{a}'}{}_a \left(\Lambda^{-1}\right)^a{}_c(\hat{\gamma}, \gamma; \tau_1) \Lambda^c{}_b(\hat{\gamma}, \gamma; \tau_3)\, {}_{\hat{\gamma}} g^b{}_{\hat{a}'''} g^{\hat{a}'''}{}_{a'''}. \tag{4.183} $$

Putting this all together, one finds that

$$ \begin{aligned} \Delta\xi^{a'}_{\mathrm{CD}}(\gamma, \bar{\gamma}) = g^{a'}{}_{\hat{a}'} \Bigg\{ & \Delta\xi^{\hat{a}'}_{\mathrm{CD}}(\hat{\gamma}, \hat{\bar{\gamma}}) + g^{\hat{a}'}{}_{\hat{\bar{a}}'} \xi^{\hat{\bar{a}}'}(\hat{\bar{\gamma}}, \bar{\gamma}) - \xi^{\hat{a}'}(\hat{\gamma}, \gamma) \\ & + {}_{\hat{\gamma}} g^{\hat{a}'}{}_a \left[\left(\Lambda^{-1}\right)^a{}_b(\hat{\gamma}, \gamma; \tau_1) - \delta^a{}_b\right] \left[\xi^b + (\tau_1 - \tau_0)\dot{\xi}^b\right] \\ & + {}_{\hat{\gamma}} g^{\hat{a}'}{}_a \left(\Lambda^{-1}\right)^a{}_c(\hat{\gamma}, \gamma; \tau_1) \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 \int_{\tau_0}^{\tau_2} \mathrm{d}\tau_3 \Lambda^c{}_b(\hat{\gamma}, \gamma; \tau_3) \\ & \qquad \times {}_{\hat{\gamma}} g^b{}_{\hat{b}'''} g^{\hat{b}'''}{}_{b'''} \left(g^{b'''}{}_{\bar{b}'''} \ddot{\bar{\gamma}}^{\bar{b}'''} - \ddot{\gamma}^{b'''}\right) \Bigg\}. \end{aligned} \tag{4.184} $$

The terms with holonomies in this expression are given by equations (4.179) and (4.180). The remaining terms are determined by noting that

$$ \xi^{\hat{a}'}(\hat{\gamma}, \gamma) = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 (\tau_1 - \tau_2)\, {}_{\hat{\gamma}} H^{\hat{a}'}{}_{\hat{a}''} g^{\hat{a}''}{}_{a''} \ddot{\gamma}^{a''}, \tag{4.185} $$

which can be proven using equation (4.108). An analogous statement holds for $\xi^{\hat{\bar{a}}'}(\hat{\bar{\gamma}}, \bar{\gamma})$.

4.3 Exact Plane Wave Spacetimes

Given results in arbitrary spacetimes with a flat-to-flat transition, we now consider our three persistent gravitational wave observables in plane wave spacetimes. A flat-to-flat transition in such spacetimes can be easily arranged by noting that these spacetimes possess a gravitational wave amplitude that is a function of a coordinate u (representing the gravitational wave phase) and proportional to the curvature of the spacetime. By simply requiring that this wave amplitude vanish outside of a given range of values of u, one obtains two regions where the curvature vanishes, sandwiching a region that has nonzero curvature.

A key result of this section is that the curve deviation and holonomy observables, when considered for geodesic curves, can be determined *exactly* in plane wave spacetimes, to all orders in initial separation and relative velocity. In particular, they can be written in terms of two sets of functions, the *transverse Jacobi propagators*, and their first derivatives. As their name suggests, these functions are the transverse components of the Jacobi propagators defined in section 4.2.1.3 above, which have been extensively studied in these spacetimes [90, 88, 89]. The information needed to construct the transverse Jacobi propagators can be obtained by measuring the displacement memory (leading and subleading [67]) in these spacetimes. It is known that other quantities in plane wave spacetimes, such as solutions to the geodesic equation, can be written in terms of the transverse Jacobi propagators as well [90]. The transverse Jacobi propagators and their derivatives form a set of three (and not four) independent matrix functions, due to a constraint arising from equation (4.86).

Some of our observables we consider only perturbatively, and not exactly. The first of these are our curve deviation and holonomy observables for nongeodesic curves. We find that these observables can be expressed as time integrals involving the transverse Jacobi propagators, but *cannot* be expressed locally in time in terms of these propagators and their time derivatives, as they can be for geodesic curves. A rough argument for this is that these observables can be written as integrals involving the product of the transverse Jacobi propagators and a given, but arbitrary acceleration vector. The other observable we calculate perturbatively is the persistent observable arising from a spinning test particle. Here again, it does not seem possible to express this observable locally in time in terms of products and derivatives of transverse Jacobi propagators, likely because this observable is *also* defined in terms of an accelerating curve. Observables which cannot be written locally in time in terms of products and derivatives of transverse Jacobi propagators measure features of the gravitational waves that are independent of the leading and subleading displacement memory and the relative velocity observables.

The layout of this section is as follows. First, we consider how nonlinear corrections are qualitatively different from first-order results, motivating the study of persistent observables in exact, as opposed to linearized, plane waves. To do so, we consider a toy model that shares some features with geodesic deviation in plane wave spacetimes. We then review exact plane wave spacetimes, studying in particular the solutions to the geodesic equation, their five-dimensional space of isometries, and the values of the various bitensors which we introduced in section 4.2.1. We conclude this section with our results, namely the computation of the curve deviation, holonomy, and spinning test particle observables.

4.3.1 Motivation

The effects we will compute in this section will be nonlinear in the amplitude of the gravitational waves. Gravitational waves produced by astrophysical sources, however, will be weak when the waves have reached any detector, so effects that are nonlinear in the amplitude of the gravitational wave are not expected to be detectable by current detectors.[3] Nevertheless, these effects are qualitatively different from linear effects, and therefore interesting in their own right. There may also be regimes in which they are detectable by future detectors.

To illustrate the types of distinctive effects that can arise in persistent observables beyond the linearized approximation, we now discuss a toy model of geodesic deviation. Consider the following differential equation for a function $\xi(u)$:

$$\ddot{\xi}(u) = \epsilon \ddot{f}(u)\xi(u), \tag{4.186}$$

where $\epsilon \ll 1$, and dots denote derivatives with respect to u. This is a scalar version of the geodesic deviation equation, where ξ is the separation between two observers and ϵf is the equivalent of the gravitational wave strain amplitude. Consider the analogue of a burst of gravitational waves that occurs between $u = 0$ and $u = U$, where $\ddot{f}(u) = 0$ for all $u < 0$ and $u > U$. For simplicity, set $f(u) = 0$ for $u < 0$; in general, this will imply that $f(u) \neq 0$ for $u > U$. The solution for $\xi(u)$ at some time $u > 0$ is then given by

$$\xi(u) = a(u)\xi(0) + b(u)\dot{\xi}(0), \tag{4.187}$$

where

$$a(0) = \dot{b}(0) = 1, \qquad \dot{a}(0) = b(0) = 0. \tag{4.188}$$

Therefore, to second order in ϵ,

$$a(u) = 1 + \epsilon f(u) + \frac{1}{2}\epsilon^2[f(u)]^2 - \epsilon^2 \int_0^u \mathrm{d}u' \int_0^{u'} \mathrm{d}u''[\dot{f}(u'')]^2 + O(\epsilon^3). \tag{4.189}$$

[3] Note that the nonlinear memory effect [53] is *not* nonlinear in the amplitude of the gravitational wave at the detector; rather, it arises from a nonlinearity in Einstein's equations in asymptotically flat spacetimes. It is much more likely to be detected by current and future gravitational wave detectors [112, 102, 97, 34].

The counterpart of the first-order memory in this case is given by the term in (4.189) linear in ϵ [that is, $f(u)$ after the burst]. This function is at most linear in u, since $\ddot{f}(u)$ is zero at late times. However, even if this first-order memory is zero [that is, if $f(u) = 0$ for $u > U$], the second-order memory is nonzero, and would, in general, grow linearly with time:

$$a(u > U) = \epsilon^2(Cu + D) + O(\epsilon^3), \tag{4.190}$$

where

$$C \equiv -\int_0^\infty \mathrm{d}u' [\dot{f}(u')]^2 \geq 0, \qquad D \equiv \int_0^\infty \mathrm{d}u' \int_{u'}^\infty \mathrm{d}u'' [\dot{f}(u'')]^2 \geq 0, \tag{4.191}$$

since $[\dot{f}(u)]^2 \geq 0$, for all u; moreover, equality can only hold in equation (4.191) if $f(u) = 0$ for all u. Since the coefficient C is nonzero, observers in this simplified model would have a relative velocity after the burst: at second order all nontrivial solutions must have $\dot{a}(u) \neq 0$ after the burst. At first order, there is no such restriction on the final relative velocity, so first- and second-order calculations yield qualitatively different results. While equation (4.186) is only a simplified model of geodesic deviation, the explicit discussion given in section 4.3.2.3 is qualitatively similar. For example, nonlinear plane wave spacetimes always have a nonzero relative velocity after a burst [36, 89, 189].

Another motivation for considering nonlinear plane wave spacetimes is as follows. Our persistent observables are "degenerate" in the linearized, plane wave limit, in the sense that they can be written in terms of only three functions (one, two, and three time integrals of the Riemann tensor) in the case where the observers are unaccelerated (see table 4.1), even though the form of the observables allows them to have more degrees of freedom than three functions possess. This implies that while our observables can encode a wide range of qualitatively different physical effects, the effects are all determined by the same, limited set of properties of the gravitational wave. One might expect that at higher order these degeneracies are broken. However, we instead find that these degeneracies (or linear relationships between observables) are replaced with nonlinear relationships between observables.

An example of such a nonlinear relationship occurs in the toy model (4.186): it can be shown from equation (4.186) that the Wronskian

$$W = a(u)\dot{b}(u) - \dot{a}(u)b(u) \tag{4.192}$$

must be conserved, and by equation (4.188), we have $W = 1$. This holds to all orders in ϵ; however, one can use equation (4.188) to show that

$$\dot{a}(u) = O(\epsilon), \quad a(u) = 1 + \int_0^u \mathrm{d}u' \dot{a}(u'), \quad b(u) = u + O(\epsilon), \tag{4.193}$$

from which equation (4.192) becomes

$$\dot{b}(u) = 1 - \int_0^u \mathrm{d}u' [\dot{a}(u') - \dot{a}(u)] + O(\epsilon^2). \tag{4.194}$$

The quantity (4.194) is an example of a way of writing observables in terms of one another that holds at first order in the curvature (corresponding to a degeneracy), but not at higher orders. This example, moreover, shows that some relationships that hold at first order are approximations to fully nonlinear relationships between observables. Much of this section focuses upon finding and understanding these nonlinear relationships.

4.3.2 Review of exact plane wave spacetimes

In this section, we review properties of exact, nonlinear plane wave spacetimes. These are spacetimes with metrics that can be written, in Brinkmann coordinates (u, v, x^1, x^2) [42], as

$$\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + \underline{\mathcal{A}}_{ij}(u)x^i x^j \mathrm{d}u^2 + \mathrm{d}x^i \mathrm{d}x^j \delta_{ij}, \tag{4.195}$$

where u is the phase of the gravitational wave, and $\underline{\mathcal{A}}_{ij}(u)$ is the wave profile.[4] The particular signs and constant factors that have been chosen in this metric are the same as those in [90]. Our convention for tensor components in Brinkmann coordinates is that we use u and v as indices for u and v components, and we use lowercase Latin letters from the middle of the alphabet (i, j, etc.) for the remaining two components, which we will call the *transverse* components. When considering generic components in Brinkmann coordinates, we use lowercase Greek letters from the middle of the alphabet (μ, ν, etc.).[5] For these component indices, we use the Einstein summation convention. Tensors which are only nonzero in their transverse (i, j, etc.) components we denote with underlines, and refer to as being transverse.

[4]Another coordinate system, Rosen coordinates [140], is often used in these spacetimes. This coordinate system is the nonlinear generalization of TT gauge for linearized gravity; see, for example, [89] for more details.

[5]Note that we have used Greek letters as other types of indices in sections 4.1 and 4.2. I apologize for any notational confusion that may arise from this.

We now list several basic features of these spacetimes which we will need (for a review, see [60]). The first is the existence of a null vector field ℓ^a which is covariantly constant:

$$\nabla_a \ell^b = 0. \tag{4.196}$$

In terms of Brinkmann coordinates, this vector field is given by

$$\ell^a \equiv -(\partial_v)^a \tag{4.197}$$

(note that our convention for ℓ^a is that of [60], which differs from that of [90] by a sign). We also define an antisymmetric tensor

$$\underline{\epsilon}_{ab} \equiv 2(\mathrm{d}x^1)_{[a}(\mathrm{d}x^2)_{b]}. \tag{4.198}$$

This tensor is transverse, and is a volume form on surfaces of constant u and v. Finally, the Riemann tensor in plane wave spacetimes is given by

$$R_{abcd} = 4\ell_{[a}\underline{\mathcal{A}}_{b][c}\ell_{d]}, \tag{4.199}$$

where

$$\underline{\mathcal{A}}_{ab} \equiv \underline{\mathcal{A}}_{ij}(u)(\mathrm{d}x^i)_a(\mathrm{d}x^j)_b. \tag{4.200}$$

It then follows from $\underline{\mathcal{A}}_{ab}\ell^b = 0$ that the only constraint from Einstein's equations is that

$$T_{ab} = -8\pi\underline{\mathcal{A}}^c{}_c\ell_a\ell_b. \tag{4.201}$$

Therefore, in vacuum, $\underline{\mathcal{A}}^a{}_a = 0$.

4.3.2.1 Geodesics and symmetries

We now discuss the solution of the geodesic and Killing equations in plane wave spacetimes. Consider a geodesic γ that, as in previous sections, we affinely parametrize by τ. At a given value of τ, we denote the coordinates of $\gamma(\tau)$ by u, v, and $x^i(\tau)$, and at τ', we denote the coordinates by u', v', and $x^i(\tau')$.[6]

We define the parameter

$$\chi \equiv \dot{\gamma}^a\ell_a, \tag{4.202}$$

[6] Note that $x^i(\tau')$ lacks a prime on the index i; this notation will be justified in section 4.3.2.2.

which is conserved along the geodesic γ by equation (4.196). This implies that

$$u' = u + \chi(\tau' - \tau). \tag{4.203}$$

Geodesics can be classified by whether or not χ vanishes. For the case $\chi = 0$, the geodesic lies entirely within a surface of constant u, and one can show that

$$\ddot{x}^i(\tau) = 0, \qquad \ddot{v} = 0; \tag{4.204}$$

therefore, the solutions of the geodesic equation are linear functions of τ. For the case $\chi \neq 0$, the geodesic equation for $x^i(\tau)$ is given by

$$\ddot{x}^i(\tau) = \chi^2 \underline{\mathcal{A}}^i{}_j(u) x^j(\tau), \tag{4.205}$$

which has nontrivial solutions.

The solutions to equation (4.205) can be written in terms of two functions of u and u', $\underline{K}^i{}_j(u', u)$ and $\underline{H}^i{}_j(u', u)$, that satisfy the differential equations

$$\partial^2_{u'} \underline{K}^i{}_j(u', u) = \underline{\mathcal{A}}^i{}_k(u') \underline{K}^k{}_j(u', u), \tag{4.206a}$$

$$\partial^2_{u'} [(u' - u)\underline{H}^i{}_j(u', u)] = (u' - u)\underline{\mathcal{A}}^i{}_k(u') \underline{H}^k{}_j(u', u), \tag{4.206b}$$

with the boundary conditions

$$\underline{K}^i{}_j(u, u) = \underline{H}^i{}_j(u, u) = \delta^i{}_j, \tag{4.207a}$$

$$\partial_{u'} \underline{K}^i{}_j(u', u)\big|_{u'=u} = \partial_{u'} \underline{H}^i{}_j(u', u)\big|_{u'=u} = 0 \tag{4.207b}$$

(see, for example, [90]). We call these functions the *transverse Jacobi propagators*, since they are related to the transverse components of the Jacobi propagators (as we will discuss in section 4.3.2.2). When we say that something in plane wave spacetimes is known "exactly," we mean that it can be written in terms of $\underline{K}^i{}_j(u', u)$ and $\underline{H}^i{}_j(u', u)$. The solution to equation (4.205), in terms of the transverse Jacobi propagators, is

$$x^i(\tau') = \underline{K}^i{}_j(u', u) x^j(\tau) + (\tau' - \tau)\underline{H}^i{}_j(u', u)\dot{x}^j(\tau), \tag{4.208}$$

where on the right-hand side u' and u are determined from τ and τ' by equation (4.203).

Next, to solve for v' when $\chi \neq 0$, for convenience we assume that γ is timelike. Note that

$$\psi^a \equiv -2v\ell^a + x^i(\partial_i)^a \tag{4.209}$$

is a proper homothety, satisfying $\mathcal{L}_\psi g_{ab} = 2g_{ab}$ (see, for example, [115]). As a consequence of this [128], it follows that $\dot\gamma^a\psi_a + \tau$ is conserved along γ, so one can write v' in terms of the coordinate v of $\gamma(\tau)$:

$$v' = v - \frac{1}{2\chi}\left[x^i(\tau')\dot x_i(\tau') - x^i(\tau)\dot x_i(\tau) + (\tau' - \tau)\right]. \tag{4.210}$$

In the above equation, one could use the values of $x^i(\tau')$ and $\dot x^i(\tau')$ that were determined in equation (4.208) in order to write everything in terms of τ, τ', transverse Jacobi propagators, and initial data. This equation is, moreover, consistent with the normalization $\dot\gamma^a\dot\gamma_a = -1$, which implies that

$$\dot\gamma^a = \chi(\partial_u)^a + \dot x^i(\tau)(\partial_i)^a - \frac{1}{2\chi}\left[1 + \dot x^i(\tau)\dot x_i(\tau) + \underline{\mathcal{A}}_{ij}(u)x^i(\tau)x^j(\tau)\right]\ell^a. \tag{4.211}$$

The quantities $\underline{K}^i{}_j(u',u)$ and $\underline{H}^i{}_j(u',u)$ are also useful for finding Killing vectors in plane wave spacetimes [90]. Plane wave spacetimes possess a five-dimensional space of Killing vectors, which is spanned by ℓ^a [by equation (4.196)], along with a four-parameter family of Killing vector fields that are orthogonal to ℓ^a [60]. We denote a member of this family by

$$\Xi^a \equiv -x^i\dot{\underline{\Xi}}_i(u)\ell^a + \underline{\Xi}^i(u)(\partial_i)^a, \tag{4.212}$$

where the function $\underline{\Xi}^i(u)$ is any solution to

$$\ddot{\underline{\Xi}}^i(u) = \underline{\mathcal{A}}^i{}_j(u)\underline{\Xi}^j(u). \tag{4.213}$$

The value of this function at any initial phase u_0 determines its values at any other u:

$$\underline{\Xi}^i(u) = \underline{K}^i{}_j(u,u_0)\underline{\Xi}^j(u_0) + (u - u_0)\underline{H}^i{}_j(u,u_0)\dot{\underline{\Xi}}^j(u_0). \tag{4.214}$$

Since $\underline{\Xi}^i(u_0)$ and $\dot{\underline{\Xi}}^i(u_0)$ are four numbers, the space of Killing vectors of the form (4.212) is four-dimensional.

Finally, we list a few useful properties of the transverse Jacobi propagators $\underline{K}^i{}_j(u',u)$ and $\underline{H}^i{}_j(u',u)$ that are outlined in, for example, [88]. First, equation (4.206) implies [7]

$$\underline{K}_k{}^i(u',u)\partial_{u'}\left[(u'-u)\underline{H}^k{}_j(u',u)\right] - (u'-u)\underline{H}^k{}_j(u',u)\partial_{u'}\underline{K}_k{}^i(u',u) = \delta^i{}_j. \tag{4.215}$$

[7] For arbitrary solutions $\underline{K}^i{}_j(u',u)$ and $\underline{H}^i{}_j(u',u)$ to equations (4.206) (that is, ignoring boundary conditions), we note that the quantity in equation (4.215) is independent of u and u'. One can think of this quantity as a conserved symplectic form on the space of solutions to equations (4.206) [84], and equations (4.206) form a Hamiltonian system [170].

This relationship is an analogue of equation (4.192), and shows that there are only *three* independent quantities amongst $\underline{K}^i{}_j(u', u)$, $\underline{H}^i{}_j(u', u)$, $\partial_{u'}\underline{K}^i{}_j(u', u)$, and $\partial_{u'}\underline{H}^i{}_j(u', u)$. One can also show the following relationships hold when these two propagators' arguments are switched:

$$\underline{H}^i{}_j(u', u) = \underline{H}_j{}^i(u, u'), \qquad \partial_{u'}\underline{K}^i{}_j(u', u) = -\partial_u\underline{K}_j{}^i(u, u'). \tag{4.216}$$

Finally, using the fact that derivatives of the transverse Jacobi propagators with respect to their second argument also must satisfy equation (4.206), one has that

$$\partial_u\underline{K}^i{}_j(u', u) = -(u' - u)\underline{H}^i{}_k(u', u)\underline{\mathcal{A}}^k{}_j(u), \tag{4.217a}$$

$$\partial_u\left[(u' - u)\underline{H}^i{}_j(u', u)\right] = -\underline{K}^i{}_j(u', u). \tag{4.217b}$$

These identities are quite useful for deriving the results in section 4.3.3.

4.3.2.2 Parallel and Jacobi propagators

In this section, we provide explicit expressions for the parallel and Jacobi propagators discussed in sections 4.2.1.2 and 4.2.1.3. These bitensors are most naturally expressed in terms of the transverse Jacobi propagators defined in section 4.3.2.1 above.

We first make the following remark: by computing the spin coefficients in these spacetimes, one can show that

$$\nabla_a(\mathrm{d}x^i)_b = \underline{\mathcal{A}}^i{}_j(u)x^j\ell_a\ell_b. \tag{4.218}$$

This implies that the transverse components of the parallel propagator is always trivial:

$${}_\gamma g^{i'}{}_i = (\partial_j)^{i'}(\mathrm{d}x^j)_i. \tag{4.219}$$

To simplify expressions in this chapter, we will no longer annotate the transverse indices i, j, etc. with primes in our expressions in Brinkmann coordinates, since distinguishing between primed and unprimed components is not necessary in view of equation (4.219). However, since these indices no longer indicate the point at which the bitensor is being evaluated, we will explicitly indicate the dependence on this point, which for many of the bitensors will be a dependence on proper time or u. For example, instead of writing ${}_\gamma K^{i'}{}_i$, we will write ${}_\gamma K^i{}_j(\tau', \tau)$, and ${}_\gamma K^{i'}{}_u$ will be written as

${}_\gamma K^i{}_u(\tau')$. This notation is consistent with the fact that we referred to the x^i coordinates of $\gamma(\tau)$ and $\gamma(\tau')$ by $x^i(\tau)$ and $x^i(\tau')$, respectively, in section 4.3.2.1.

The values of the parallel and Jacobi propagators are different based on whether the parameter χ is zero or nonzero. In this chapter, we will only need the values of the parallel propagator either when $\chi = 0$ and $\underline{\mathcal{A}}_{ij}(u) = 0$, or when $\chi \neq 0$. The first of these two cases is trivial, as spacetime is flat:

$$ {}_\gamma g^{a'}{}_a = {}_\gamma K^{a'}{}_a = {}_\gamma H^{a'}{}_a = (\partial_\mu)^{a'} (\mathrm{d}x^\mu)_a, \tag{4.220} $$

where, as mentioned above, x^μ refers to the μth Brinkmann coordinate.

We now find the values of the parallel and Jacobi propagators for the case $\chi \neq 0$ using a method similar to that of [117], which was used to determine the parallel propagator in the Kerr spacetime. This method is given by finding convenient choices of the bases $(e_\alpha)^a$, $({}_K e_\alpha)^a$, and $({}_H e_\alpha)^a$ introduced in sections 4.2.1.2 and 4.2.1.3. Two of the basis elements are given by $\dot\gamma^a$ and ℓ^a, which automatically satisfy equations (4.52) and (4.68):

$$ (e_0)^{a'} = ({}_K e_0)^{a'} = ({}_H e_0)^{a'} = \dot\gamma^{a'}, \tag{4.221a} $$

$$ (e_3)^{a'} = ({}_K e_3)^{a'} = ({}_H e_3)^{a'} = \ell^{a'}. \tag{4.221b} $$

For the other two, we first define a projection operator

$$ {}_\gamma P^a{}_b = \delta^a{}_b - \frac{1}{\chi} \ell^a \dot\gamma_b, \tag{4.222} $$

and then write

$$ (e_i)^{a'} = {}_\gamma P^{a'}{}_{b'} (\partial_i)^{b'}, \qquad ({}_K e_i)^{a'} = {}_\gamma P^{a'}{}_{b'} (\partial_j)^{b'} K^j{}_i(u', u), \qquad ({}_H e_i)^{a'} = {}_\gamma P^{a'}{}_{b'} (\partial_j)^{b'} H^j{}_i(u', u). \tag{4.223} $$

At $\gamma(\tau)$, all of these expressions agree, and moreover one can readily show that they obey equations (4.52) and (4.68). The dual basis at $\gamma(\tau)$ is given by

$$ (\omega^0)_a = \frac{1}{\chi} \ell_a, \tag{4.224a} $$

$$ (\omega^3)_a = \frac{1}{\chi} \left(\dot\gamma_a + \frac{1}{\chi} \ell_a \right), \tag{4.224b} $$

$$ (\omega^i)_a = (\mathrm{d}x^i)_b \, {}_\gamma P_a{}^b. \tag{4.224c} $$

Here, we only list the dual basis for $(e_i)^a$, since the bases agree at $\gamma(\tau)$.

When $\chi \neq 0$ and γ is timelike, the nonzero components of the parallel and Jacobi propagators are therefore [90]

$$\begin{aligned}
{}_\gamma g^{u'}{}_u &= {}_\gamma K^{u'}{}_u = {}_\gamma H^{u'}{}_u = 1, &\text{(4.225a)}\\
{}_\gamma g^{v'}{}_v &= {}_\gamma K^{v'}{}_v = {}_\gamma H^{v'}{}_v = 1, &\text{(4.225b)}\\
{}_\gamma g^{i}{}_j(\tau',\tau) &= \delta^i{}_j, &\text{(4.225c)}\\
{}_\gamma g^{i}{}_u(\tau') &= \frac{1}{\chi}\left[\dot{x}^i(\tau') - \dot{x}^i(\tau)\right], &\text{(4.225d)}\\
{}_\gamma g^{v'}{}_i(\tau) &= \frac{1}{\chi}\left[\dot{x}_i(\tau') - \dot{x}_i(\tau)\right], &\text{(4.225e)}\\
{}_\gamma g^{v'}{}_u &= \frac{1}{2\chi^2}\Big\{\left[\dot{x}^i(\tau') - \dot{x}^i(\tau)\right]\left[\dot{x}_i(\tau') - \dot{x}_i(\tau)\right] \\
&\qquad + \chi\left[\underline{\mathcal{A}}_{ij}(u')x^i(\tau')x^j(\tau') - \underline{\mathcal{A}}_{ij}(u)x^i(\tau)x^j(\tau)\right]\Big\}, &\text{(4.225f)}\\
{}_\gamma K^{i}{}_j(\tau',\tau) &= \underline{K}^i{}_j(u',u), &\text{(4.225g)}\\
{}_\gamma K^{i}{}_u(\tau') &= \frac{1}{\chi}\left[\dot{x}^i(\tau') - \underline{K}^i{}_j(u',u)\dot{x}^j(\tau)\right], &\text{(4.225h)}\\
{}_\gamma K^{v'}{}_i(\tau) &= \frac{1}{\chi}\left[\dot{x}_j(\tau')\underline{K}^j{}_i(u',u) - \dot{x}_i(\tau)\right], &\text{(4.225i)}\\
{}_\gamma K^{v'}{}_u &= \frac{1}{2\chi^2}\Big\{\dot{x}^i(\tau')\dot{x}_i(\tau') + \dot{x}^i(\tau)\dot{x}_i(\tau) - 2\dot{x}_i(\tau')\underline{K}^i{}_j(u',u)\dot{x}^j(\tau) \\
&\qquad + \chi\left[\underline{\mathcal{A}}_{ij}(u')x^i(\tau')x^j(\tau') - \underline{\mathcal{A}}_{ij}(u)x^i(\tau)x^j(\tau)\right]\Big\}, &\text{(4.225j)}\\
{}_\gamma H^{i}{}_j(\tau',\tau) &= \underline{H}^i{}_j(u',u), &\text{(4.225k)}\\
{}_\gamma H^{i}{}_u(\tau') &= \frac{1}{\chi}\left[\dot{x}^i(\tau') - \underline{H}^i{}_j(u',u)\dot{x}^j(\tau)\right], &\text{(4.225l)}\\
{}_\gamma H^{v'}{}_i(\tau) &= \frac{1}{\chi}\left[\dot{x}_j(\tau')\underline{H}^j{}_i(u',u) - \dot{x}_i(\tau)\right], &\text{(4.225m)}\\
{}_\gamma H^{v'}{}_u &= \frac{1}{2\chi^2}\Big\{\dot{x}^i(\tau')\dot{x}_i(\tau') + \dot{x}^i(\tau)\dot{x}_i(\tau) - 2\dot{x}_i(\tau')\underline{H}^i{}_j(u',u)\dot{x}^j(\tau) \\
&\qquad + \chi\left[\underline{\mathcal{A}}_{ij}(u')x^i(\tau')x^j(\tau') - \underline{\mathcal{A}}_{ij}(u)x^i(\tau)x^j(\tau)\right]\Big\}. &\text{(4.225n)}
\end{aligned}$$

As in equation (4.208), u' and u on the right-hand sides of these equations are functions of τ' and τ by equation (4.203). Note also that we have written the expressions in equations (4.225) in terms of $x^i(\tau')$ and $\dot{x}^i(\tau')$, which can be expressed in terms of $x^i(\tau)$ and $\dot{x}^i(\tau)$ using equation (4.208). One can easily show that these results agree with [90], in which $\chi = 1$.

4.3.2.3 Second-order transverse Jacobi propagators

We now compute general expressions for the transverse Jacobi propagators to second order in the curvature, which have been previously computed in [89]. In the context of an arbitrary plane wave spacetime, one can write down perturbative expansions of the transverse Jacobi propagators in powers of $\underline{\mathcal{A}}^i{}_j(u)$:

$$\underline{K}^i{}_j(u', u) = \sum_{n=0}^{\infty} {}^{(n)}\underline{K}^i{}_j(u', u), \qquad \underline{H}^i{}_j(u', u) = \sum_{n=0}^{\infty} {}^{(n)}\underline{H}^i{}_j(u', u). \tag{4.226}$$

At zeroth order, from the boundary conditions in equation (4.207), the transverse Jacobi propagators are

$$ {}^{(0)}\underline{K}^i{}_j(u', u) = {}^{(0)}\underline{H}^i{}_j(u', u) = \delta^i{}_j. \tag{4.227}$$

Higher-order terms in this expansion are then obtained by solving equations (4.206) and (4.207) iteratively. At first order, the propagators are given by

$$ {}^{(1)}\underline{K}^i{}_j(u', u) = \int_u^{u'} \mathrm{d}u'' \int_u^{u''} \mathrm{d}u''' \underline{\mathcal{A}}^i{}_j(u'''), \tag{4.228a}$$

$$ {}^{(1)}\underline{H}^i{}_j(u', u) = \int_u^{u'} \mathrm{d}u'' \int_u^{u''} \mathrm{d}u''' \frac{u''' - u}{u' - u} \underline{\mathcal{A}}^i{}_j(u'''). \tag{4.228b}$$

We write all higher-order corrections in terms of these first-order terms and their derivatives, as they provide a particularly convenient way of representing these results. Note, however, that there is a certain amount of freedom in how we write second-order terms, because of the truncation of the identity (4.215) at first order. As such, there are different ways of writing the first- and second-order results in this section, depending upon whether one uses all four of ${}^{(1)}\underline{K}^i{}_j(u', u)$, ${}^{(1)}\underline{H}^i{}_j(u', u)$, $\partial_{u'}\, {}^{(1)}\underline{K}^i{}_j(u', u)$, and $\partial_{u'}\, {}^{(1)}\underline{H}^i{}_j(u', u)$, or some subset of three. As it results in relatively compact equations, we use all four.

Continuing to second order, one can show (by an integration by parts) that

$$\begin{aligned}
{}^{(2)}\underline{K}^i{}_j(u',u) = {}& \frac{1}{2}\,{}^{(1)}\underline{K}^i{}_k(u',u)\,{}^{(1)}\underline{K}^k{}_j(u',u) \\
& - \int_u^{u'} \mathrm{d}u'' \int_u^{u''} \mathrm{d}u''' \Big\{ \partial_{u'''}\,{}^{(1)}\underline{K}^i{}_k(u''',u)\partial_{u'''}\,{}^{(1)}\underline{K}^k{}_j(u''',u) \\
& \qquad - \frac{1}{2}\big[\underline{\mathcal{A}}(u'''),\,{}^{(1)}\underline{K}(u''',u)\big]^i{}_j \Big\},
\end{aligned} \tag{4.229a}$$

$$\begin{aligned}
(u'-u)\,{}^{(2)}\underline{H}^i{}_j(u',u) = {}& \frac{1}{2}(u'-u)\,{}^{(1)}\underline{H}^i{}_k(u',u)\,{}^{(1)}\underline{H}^k{}_j(u',u) \\
& - \int_u^{u'} \mathrm{d}u'' \int_u^{u''} \mathrm{d}u''' \Big\{ (u'''-u)\partial_{u'''}\,{}^{(1)}\underline{H}^i{}_k(u''',u)\partial_{u'''}\,{}^{(1)}\underline{H}^k{}_j(u''',u) \\
& \qquad - \frac{1}{2}(u'''-u)\big[\underline{\mathcal{A}}(u'''),\,{}^{(1)}\underline{H}(u''',u)\big]^i{}_j \Big\},
\end{aligned} \tag{4.229b}$$

where the commutator $[A,B]^a{}_b$ is given by

$$[A,B]^a{}_b \equiv A^a{}_c B^c{}_b - B^a{}_c A^c{}_b. \tag{4.230}$$

Note that there are two types of terms that appear in equations (4.229) at second order. The first are terms that are merely squares of the final values of the first-order terms; these are the first terms in equations (4.229a) and (4.229b). The other two terms in both equations are qualitatively different at second order. They are generically nonzero, even when the final values of the first-order terms vanish, as they depend on integrals of squares of first-order quantities throughout the curved region. These terms are analogous to the fourth term in the solution (4.189) of the toy model for geodesic deviation in section 4.3.1.

The various terms at second order are also qualitatively different in the following sense. Assuming a vacuum plane wave, one has that $\underline{\mathcal{A}}^i{}_j(u)$ is traceless, and so ${}^{(1)}\underline{K}^i{}_j(u',u)$ and ${}^{(1)}\underline{H}^i{}_j(u',u)$ are traceless as well. Thus, we find that the first two terms in (4.229a) and (4.229b) are pure trace, as they are squares of 2×2 symmetric, trace-free matrices, and that the third terms are antisymmetric (for a proof of these statements, see appendix 4.A to this chapter). Because of the existence of pure trace terms at second order, gravitational waves possess an effective "breathing" polarization mode [183] at this order [89]. Note that the third (antisymmetric) term in equation (4.229a) vanishes when the gravitational waves are linearly polarized; this effect was previously noted in [190].

4.3.2.4 Example of a plane wave spacetime

We now illustrate the general results of section 4.3.2.3 by specializing to an explicit example of a plane wave spacetime. We choose the following form of $\underline{\mathcal{A}}_{ij}(u)$: for positive ω, let[8]

$$\underline{\mathcal{A}}_{ij}(u) = \begin{cases} \epsilon\omega^2 \left[\sqrt{1-a^2}\sin(\omega u)\,{}_{+}\underline{e}_{ij} + a\sin(\omega u + \phi)\,{}_{\times}\underline{e}_{ij}\right] & 0 \le u \le 2\pi n/\omega \\ 0 & u < 0,\ u > 2\pi n/\omega \end{cases}, \tag{4.231}$$

where n is a positive integer representing the number of periods over which the gravitational wave is nonzero, and

$${}_{+}\underline{\boldsymbol{e}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad {}_{\times}\underline{\boldsymbol{e}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.232}$$

are polarization matrices corresponding to $+$ and $\times$ polarizations. Some special cases are linear polarization, where $\phi = 0$, and circular polarization, where $\phi = \pm\pi/2$ and $a = 1/\sqrt{2}$. This wave pulse also satisfies $\int_{-\infty}^{\infty} \mathrm{d}u \underline{\mathcal{A}}_{ij}(u) = 0$, which (at first order) means vanishing relative velocity at late times for observers that are initially comoving. Gravitational waves at null infinity are also frequently assumed to satisfy a condition analogous to $\int_{-\infty}^{\infty} \mathrm{d}u \underline{\mathcal{A}}_{ij}(u) = 0$.

Using the explicit wave profile in equation (4.231), we find that

$$\begin{aligned} \underline{K}^i{}_j(2\pi n/\omega, 0) = \delta^i{}_j &+ 2\pi n\epsilon \left[\sqrt{1-a^2}\,{}_{+}\underline{e}^i{}_j + \cos(\phi)\,{}_{\times}\underline{e}^i{}_j\right] \\ &- \frac{\pi n\epsilon^2}{2}\left\{\left[2\pi n + 3\sin(2\phi)a^2\right]\delta^i{}_j - 12\sin(\phi)a\sqrt{1-a^2}\underline{\epsilon}^i{}_j\right\} + O(\epsilon^3), \end{aligned} \tag{4.233a}$$

$$\begin{aligned} \underline{H}^i{}_j(2\pi n/\omega, 0) = \delta^i{}_j &- 2\epsilon\sin(\phi)a\,{}_{\times}\underline{e}^i{}_j \\ &- \frac{\epsilon^2}{2}\left[\frac{4\pi^2 n^2 + 9[\cos(2\phi) - 1] - 15}{6}\delta^i{}_j - 4\pi\sin(\phi)a\sqrt{1-a^2}n\underline{\epsilon}^i{}_j\right] + O(\epsilon^3), \end{aligned} \tag{4.233b}$$

[8] This wave profile is periodic, so in the fully nonlinear regime Floquet theory (see, for example, [116] and references therein) applies to equation (4.206) and its solutions. Although it is outside the scope of this thesis, it would be interesting to use this fact to determine regions in the parameter space of ϵ, a, and ϕ where solutions are bounded and regions in this parameter space where they are unbounded.

$$\partial_u \underline{K}^i{}_j(u,0)\big|_{u=2\pi n/\omega} = -\omega\pi n\epsilon^2\{[\cos(2\phi) - 1]a^2 + 3\}\delta^i{}_j + O(\epsilon^3), \tag{4.233c}$$

$$\begin{aligned}\partial_u \underline{H}^i{}_j(u,0)\big|_{u=2\pi n/\omega} = -\omega\epsilon\left[\sqrt{1-a^2}\,{}_+\underline{e}^i{}_j + a\frac{\pi n\cos(\phi) - \sin(\phi)}{\pi n}\,{}_\times\underline{e}^i{}_j\right] \\ - \frac{\omega\epsilon^2}{2}\left[\frac{8\pi^2 n^2 - 9a^2[2\pi\sin(2\phi)n - \cos(2\phi) + 1] + 15}{12\pi n}\delta^i{}_j\right. \\ \left. - 4\sin(\phi)a\sqrt{1-a^2}\underline{\epsilon}^i{}_j\right] + O(\epsilon^3).\end{aligned} \tag{4.233d}$$

Qualitatively, the results of section 4.3.2.3 agree with these equations: the symmetric trace-free terms (proportional to ${}_+\underline{e}^i{}_j$ and ${}_\times\underline{e}^i{}_j$) occur only at first order, and the pure trace terms (proportional to $\delta^i{}_j$) and antisymmetric terms (proportional to $\epsilon^i{}_j$) occur only at second order. As expected, equation (4.233c) implies that the final relative velocity for initially comoving observers vanishes at first order. Finally, the antisymmetric pieces only occur when the polarization is not linear ($\phi \neq 0$).

To study the long-time behavior of these solutions, consider the regime where $n \to \infty$ as $\epsilon \to 0$, with

$$n \sim \frac{1}{\epsilon^{1-\eta}}. \tag{4.234}$$

We assume $0 < \eta < 1$ so that the series (4.226) converges. In this regime, the antisymmetric terms in equations (4.233) are subleading compared to the symmetric terms.

4.3.3 Results

In this section, we compute the curve deviation, holonomy, and spinning test particle observables in the context of plane wave spacetimes. For the first two of these observables, we make the assumption that the curves γ and $\bar{\gamma}$ that occur in the definitions of these observables are geodesic. In this case, we can take advantage of the fact that the geodesic equation has exact solutions in plane wave spacetimes in terms of the transverse Jacobi propagators, as reviewed in section 4.3.2.1. Moreover, we take advantage of the known values of the parallel and Jacobi propagators in these spacetimes reviewed in section 4.3.2.2. This allows us to find expressions which are nonperturbative in the initial separation and relative velocity. In the case where these two observables involve accelerated curves, one can use these results, combined with those proven in section 4.2.3.4. The results in these cases are necessarily perturbative in the acceleration.

The spinning test particle observable, however, does not have a nonperturbative treatment, and so we use the results of section 4.2.3.3 that are perturbative in separation, specialized to the class of plane wave spacetimes. We could have used the same technique to derive perturbative results for the first two observables in plane wave spacetimes, but we did not because we already have analytic, nonperturbative results.

We now introduce two pieces of notation that are used extensively in this section. First, for given points $x^{(n)} = \gamma(\tau_n)$, we denote the coordinates of these points by u_n, v_n, and $x^i(\tau_n)$. This convention also holds for curves denoted by γ with some sort of diacritical marking above or below: we apply the same diacritical mark to the point in question as well [for example, $\bar{x}'$, referring to $\bar{\gamma}(\tau_1)$, has coordinates $\bar{u}_1$, $\bar{v}_1$, and $\bar{x}^i(\tau_1)$]. A figure showing the setup common to all persistent observables discussed in this section, with appropriate annotation for plane waves, is given in figure 4.7.

Second, many of the results in this section depend not only upon $\underline{\mathcal{A}}_{ij}(u)$ and the propagators $\underline{K}^i{}_j(u', u)$ and $\underline{H}^i{}_j(u', u)$, which are only functions of u and u', but also upon $x^i(\tau_0)$ and $\dot{x}^i(\tau_0)$. This dependence is at most polynomial for the observables which we consider. For some bitensor component $Q^{\cdots}{}_{\cdots}$ in Brinkmann coordinates (for simplicity we suppress the indices) that depends on $x^i(\tau_0)$ and $\dot{x}^i(\tau_0)$, we can write

$$Q^{\cdots}{}_{\cdots} \equiv \sum_{k,m} {}_{x^k \dot{x}^m}[Q^{\cdots}{}_{\cdots}]_{i_1 \cdots i_k j_1 \cdots j_m} x^{i_1}(\tau_0) \cdots x^{i_k}(\tau_0) \dot{x}^{j_1}(\tau_0) \cdots \dot{x}^{j_m}(\tau_0). \tag{4.235}$$

Examples of this notation occur throughout this section; for example, in equations (4.243b) and (4.243c) the quantities ${}_x[\Delta K^{v'}{}_i]_j(\tau_0)$ and ${}_{\dot{x}}[\Delta K^{v'}{}_i]_j(\tau_0)$ are coefficients in the expansion of the component $\Delta K^{v'}{}_i(\tau_0)$ of $\Delta K^{a'}{}_b$ in powers of $x^i(\tau_0)$ and $\dot{x}^i(\tau_0)$:

$$\Delta K^{v'}{}_i(\tau_0) = {}_x[\Delta K^{v'}{}_i]_j(\tau_0) x^j(\tau_0) + {}_{\dot{x}}[\Delta K^{v'}{}_i]_j(\tau_0) \dot{x}^j(\tau_0). \tag{4.236}$$

There are often relationships between the coefficients that occur in these expansions, see for example equation (4.266).

4.3.3.1 Curve deviation observable

In this section, we consider the curve deviation observable $\Delta\xi^{a'}_{\mathrm{CD}}$ defined in section 4.1.2. In plane wave spacetimes, the geodesic equation has exact solutions in terms of transverse Jacobi propagators,

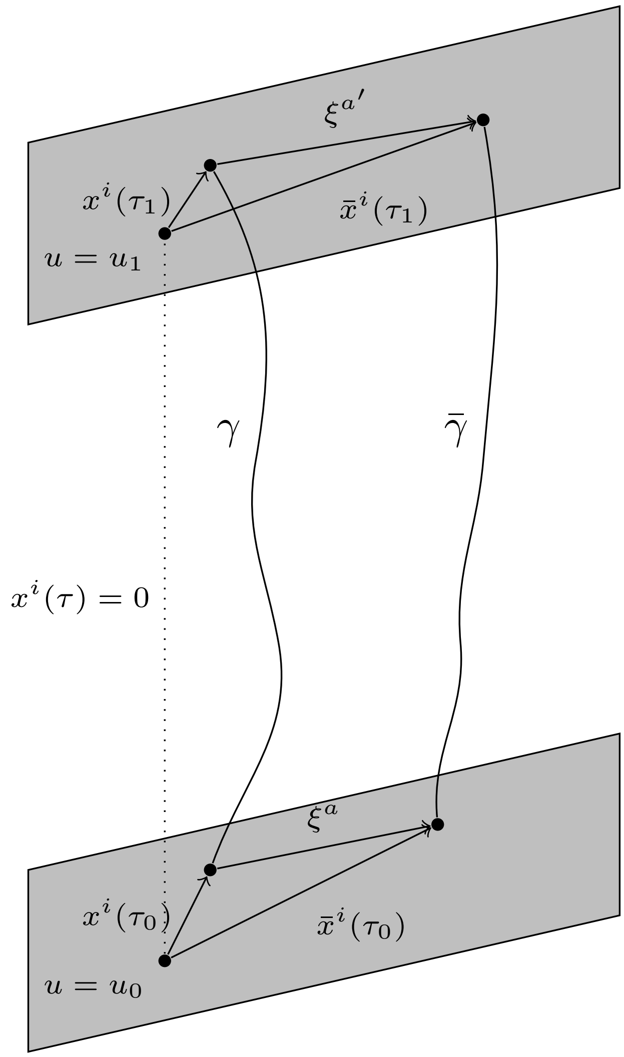

Figure 4.7: The common setup for all persistent observables discussed in this section: two timelike curves γ and $\bar{\gamma}$ that have some initial separation ξ^a at time τ_0 and final separation $\xi^{a'}$ at time τ_1. Intersecting these two curves are two planes of constant u (the v coordinate in this diagram is suppressed). The x^i coordinates of the points $\gamma(\tau_0)$, $\gamma(\tau_1)$, $\bar{\gamma}(\tau_0)$, and $\bar{\gamma}(\tau_1)$ are also shown in this diagram.

and moreover the parallel propagators are known along geodesics. As such, curve deviation, at least restricting to the case where there is no acceleration for either curve, will have an exact solution, instead of a perturbative solution in the separation of the two particles.

In order to compute the observables in equation (4.8), we need the separation vector in the flat regions of the plane wave spacetime in Brinkmann coordinates. Before the burst, this separation vector is given by

$$\xi^a = (\bar{u}_0 - u_0)(\partial_u)^a + [\bar{x}^i(\tau_0) - x^i(\tau_0)](\partial_i)^a - (\bar{v}_0 - v_0)\ell^a. \tag{4.237}$$

A similar expression holds for $\xi^{a'}$, the separation vector after the burst. We also require the relative velocity before the burst. In the flat regions, $\dot{\xi}^a = g^a{}_{\bar{a}}\dot{\bar{\gamma}}^a - \dot{\gamma}^a$, and so we find that [from equation (4.211)]

$$\begin{aligned}\dot{\xi}^a = {} & (\bar{\chi} - \chi)(\partial_u)^a + \left[\dot{\bar{x}}^i(\tau_0) - \dot{x}^i(\tau_0)\right](\partial_i)^a \\ & - \left\{\frac{1}{2\bar{\chi}}\left[1 + \dot{\bar{x}}^i(\tau_0)\dot{\bar{x}}_i(\tau_0)\right] - \frac{1}{2\chi}\left[1 + \dot{x}^i(\tau_0)\dot{x}_i(\tau_0)\right]\right\}\ell^a,\end{aligned} \tag{4.238}$$

where $\bar{\chi} \equiv \dot{\bar{\gamma}}^a \ell_a$.

At this point, we note that this calculation is greatly simplified in the case where we assume that $\bar{u}_0 = u_0$ (which implies that $\xi^a \ell_a = 0$) and $\bar{\chi} = \chi$ (which implies that $\dot{\xi}^a \ell_a = 0$). In particular, this assumption about the initial data means that the exact solutions are *quadratic* in ξ^a and $\dot{\xi}^a$; in general, the results are not polynomial in $\xi^a \ell_a$ and $\dot{\xi}^a \ell_a$. Note that this assumption implies that $\bar{u}_1 = u_1$ as well, from equation (4.203). Thus, we are also associating points on the two worldlines with equal values of u, the gravitational wave phase, so this restriction could be called the *isophase correspondence*.

Taking into account these assumptions, we find that $\xi^{a'}$ is given by

$$\xi^{a'} = \left[\bar{x}^i(\tau_1) - x^i(\tau_1)\right](\partial_i)^{a'} - (\bar{v}_1 - v_1)\ell^{a'}. \tag{4.239}$$

We can determine the first term in this equation by using equation (4.208), together with equations (4.237) and (4.238):

$$\bar{x}^i(\tau_1) - x^i(\tau_1) = \underline{K}^i{}_j(u_1, u_0)\xi^j(\tau_0) + (\tau_1 - \tau_0)\underline{H}^i{}_j(u_1, u_0)\dot{\xi}^j(\tau_0). \tag{4.240}$$

For the second term, we use equations (4.210) and (4.238):

$$
\begin{aligned}
&\bar{v}_1 - v_1 - (\bar{v}_0 - v_0) \\
&\quad = \frac{1}{2\chi}\left\{\left[\bar{x}^i(\tau_1)\dot{\bar{x}}_i(\tau_1) - x^i(\tau_1)\dot{x}_i(\tau_1)\right] - \left[\bar{x}^i(\tau_0)\dot{\bar{x}}_i(\tau_0) - x^i(\tau_0)\dot{x}_i(\tau_0)\right]\right\} \\
&\quad = \underline{K}^k{}_i(u_1,u_0)\partial_{u_1}\underline{K}_{kj}(u_1,u_0)\left[\frac{1}{2}\xi^i(\tau_0)\xi^j(\tau_0) + \xi^i(\tau_0)x^j(\tau_0)\right] \\
&\qquad + \frac{1}{\chi}\left\{\underline{K}^k{}_i(u_1,u_0)\partial_{u_1}\left[(u_1-u_0)\underline{H}_{kj}(u_1,u_0)\right] - \delta_{ij}\right\}\left[\frac{1}{2}\xi^i(\tau_0)\dot{\xi}^j(\tau_0) + \xi^i(\tau_0)\dot{x}^j(\tau_0)\right] \\
&\qquad + \frac{1}{\chi}(u_1-u_0)\underline{H}^k{}_i(u_1,u_0)\partial_{u_1}\underline{K}_{kj}(u_1,u_0)\left[\frac{1}{2}\dot{\xi}^i(\tau_0)\xi^j(\tau_0) + \dot{\xi}^i(\tau_0)x^j(\tau_0)\right] \\
&\qquad + \frac{1}{\chi^2}(u_1-u_0)\underline{H}^k{}_i(u_1,u_0)\partial_{u_1}\left[(u_1-u_0)\underline{H}_{kj}(u_1,u_0)\right]\left[\frac{1}{2}\dot{\xi}^i(\tau_0)\dot{\xi}^j(\tau_0) + \dot{\xi}^i(\tau_0)\dot{x}^j(\tau_0)\right].
\end{aligned}
\tag{4.241}
$$

At this point, note that the curve deviation observable is defined as the result of geodesic deviation with the prediction in flat spacetime subtracted off. This prediction is given by

$$
\begin{aligned}
{}_\gamma g^{a'}{}_a\left[\xi^a + (\tau_1-\tau_0)\dot{\xi}^a\right] = {}& \left[\xi^i(\tau_0) + (\tau_1-\tau_0)\dot{\xi}^i(\tau_0)\right] \\
& \times\left\{(\partial_i)^{a'} - \left[\partial_{u_1}\underline{K}_{ij}(u_1,u_0)x^j(\tau_0)\right.\right. \\
& \qquad\left.\left. + \frac{1}{\chi}\left\{\partial_{u_1}\left[(u_1-u_0)\underline{H}_{ij}(u_1,u_0)\right] - \delta_{ij}\right\}\dot{x}^j(\tau_0)\right]\ell^{a'}\right\} \\
& - \left\{\bar{v}_0 - v_0 + \frac{1}{\chi^2}(u_1-u_0)\left[\frac{1}{2}\dot{\xi}^i(\tau_0)\dot{\xi}_i(\tau_0) + \dot{\xi}^i(\tau_0)\dot{x}_i(\tau_0)\right]\right\}\ell^{a'},
\end{aligned}
\tag{4.242}
$$

where we have used equation (4.225) to compute the parallel propagator. We now use the decompositions in equations (4.8) and (4.235). Since the exact solutions are quadratic in ξ^a and $\dot{\xi}^a$, we just need to write down the components of $\Delta K^{a'}{}_a$, $\Delta H^{a'}{}_a$, $L^{a'}{}_{bc}$, $N^{a'}{}_{bc}$, and $M^{a'}{}_{bc}$. The nonvanishing quantities needed to compute these bitensors (and thus, the curve deviation observable) are as follows:

$$\Delta K^i{}_j(\tau_1, \tau_0) = \underline{K}^i{}_j(u_1, u_0) - \delta^i{}_j, \tag{4.243a}$$

$$_x[\Delta K^{v'}{}_i]_j(\tau_0) = \partial_{u_1} \underline{K}_{kj}(u_1, u_0) \Delta K^k{}_i(\tau_1, \tau_0), \tag{4.243b}$$

$$_{\dot{x}}[\Delta K^{v'}{}_i]_j(\tau_0) = \frac{1}{\chi} \partial_{u_1} \left[(u_1 - u_0) \underline{H}_{kj}(u_1, u_0)\right] \Delta K^k{}_i(\tau_0), \tag{4.243c}$$

$$\Delta H^i{}_j(\tau_1, \tau_0) = \underline{H}^i{}_j(u_1, u_0) - \delta^i{}_j, \tag{4.243d}$$

$$_x[\Delta H^{v'}{}_i]_j(\tau_0) = \partial_{u_1} \underline{K}_{kj}(u_1, u_0) \Delta H^k{}_i(\tau_1, \tau_0), \tag{4.243e}$$

$$_{\dot{x}}[\Delta H^{v'}{}_i]_j(\tau_0) = \frac{1}{\chi} \partial_{u_1} \left[(u_1 - u_0) \underline{H}_{kj}(u_1, u_0)\right] \Delta H^k{}_i(\tau_0), \tag{4.243f}$$

$$L^{v'}{}_{ij}(\tau_0) = \frac{1}{2} \underline{K}_{k(i}(u_1, u_0) \partial_{u_1} \underline{K}^k{}_{j)}(u_1, u_0), \tag{4.243g}$$

$$N^{v'}{}_{ij}(\tau_0) = \frac{1}{2\chi} \left\{\partial_{u_1} \left[(u_1 - u_0) \underline{K}^k{}_i(u_1, u_0) \underline{H}_{kj}(u_1, u_0)\right] - \delta_{ij}\right\}, \tag{4.243h}$$

$$M^{v'}{}_{ij}(\tau_0) = \frac{1}{2\chi^2}(u_1 - u_0) \left\{\underline{H}_{k(i}(u_1, u_0) \partial_{u_1} \left[(u_1 - u_0) \underline{H}^k{}_{j)}(u_1, u_0)\right] - \delta_{ij}\right\}. \tag{4.243i}$$

The above equations make it clear that the curve deviation observable depends only on the transverse Jacobi propagators and their first derivatives at τ_1, with no need to integrate any additional quantities from τ_0 to τ_1.

Moreover, note that $L^i{}_{jk}(\tau_1, \tau_0)$, $N^i{}_{jk}(\tau_1, \tau_0)$, and $M^i{}_{jk}(\tau_1, \tau_0)$ all vanish. This is a consequence of the fact that geodesic deviation, in the case where the initial separation lies entirely in a surface of constant u and v, has no corrections at second order in the separation, at least for the components that also lie in this surface. Because of this property, the proper time delay observable $\dot{\gamma}_{a'} \Delta\xi^{a'}_{\mathrm{CD}}$ (described in greater detail in section 4.1.2) can be expressed as

$$\dot{\gamma}_{a'} \Delta\xi^{a'}_{\mathrm{CD}} = -\chi \left[L^{v'}{}_{ij}(\tau_0) \xi^i(\tau_0) \xi^j(\tau_0) + N^{v'}{}_{ij}(\tau_0) \xi^i(\tau_0) \dot{\xi}^j(\tau_0) + M^{v'}{}_{ij}(\tau_0) \dot{\xi}^i(\tau_0) \dot{\xi}^j(\tau_0)\right]. \tag{4.244}$$

4.3.3.2 Holonomies

We now consider the holonomy observables. In general spacetimes, we needed to expand perturbatively in the separation and relative velocity of the two curves γ and $\bar{\gamma}$; in contrast, for plane waves, these calculations can be done nonperturbatively. As such, we perform only the nonperturbative calculations in this section—the perturbative results can be obtained by simply specializing the results in section 4.2.3.2 to plane wave spacetimes using equation (4.225) for the parallel and Jacobi propagators. Note that we also continue to use the assumption that $\xi^a \ell_a = 0$ and $\dot{\xi}^a \ell_a = 0$, for simplicity.

We specialize in this section to affine transport and dual Killing transport, instead of considering arbitrary $\varkappa$ as in section 4.2.3.2. First, we consider the case of affine transport. Here, we take advantage of the fact that this holonomy [using equation (4.31)] can be written in terms of $\Lambda^a{}_b(\gamma, \bar{\gamma}; \tau_1)$ and $\Delta\chi^a(\gamma, \bar{\gamma}; \tau_1)$. Therefore, all we need in order to solve for the holonomy of affine transport is the value of the separation $\xi^{a'}$ at τ_1 and the holonomy of the metric-compatible connection around the loop.

We computed the separation $\xi^{a'}$ in section 4.3.3.1, so at this point we merely need to compute $\Lambda^a{}_b(\gamma, \bar{\gamma}; \tau_1)$. Since ℓ^a is covariantly constant, it follows that $\Omega^a{}_b(\gamma, \bar{\gamma}; \tau_1)\ell^b = 0$. After a lengthy calculation using our expressions for the parallel propagators in equation (4.225), we find that

$$\Omega^a{}_b(\gamma, \bar{\gamma}; \tau_1)(\partial_i)^b = \frac{1}{\chi}\left[\dot{\xi}_i(\tau_1) - \dot{\xi}_i(\tau_0)\right]\ell^a, \tag{4.245a}$$

$$\Omega^a{}_b(\gamma, \bar{\gamma}; \tau_1)u^b = -\frac{1}{\chi}\left[\dot{\xi}^i(\tau_1) - \dot{\xi}^i(\tau_0)\right](\partial_i)^a - \frac{1}{2\chi^2}\left[\dot{\xi}^i(\tau_1) - \dot{\xi}^i(\tau_0)\right]\left[\dot{\xi}_i(\tau_1) - \dot{\xi}_i(\tau_0)\right]\ell^a. \tag{4.245b}$$

By a lengthy calculation involving equation (4.16) we can also show that

$$\Delta\chi^i(\gamma, \bar{\gamma}; \tau_1, \tau_0) = \xi^i(\tau_0) - \xi^i(\tau_1) + (\tau_1 - \tau_0)\dot{\xi}^i(\tau_1), \tag{4.246a}$$

$$\begin{aligned}\Delta\chi^v(\gamma, \bar{\gamma}; \tau_1) = \frac{1}{\chi}\bigg\{&\left[\frac{1}{2}\dot{\xi}_i(\tau_1) - \dot{\xi}_i(\tau_0)\right]\left[\xi^i(\tau_1) - (\tau_1 - \tau_0)\dot{\xi}^i(\tau_1)\right] + \frac{1}{2}\xi^i(\tau_0)\dot{\xi}_i(\tau_0) \\ &+ \dot{x}_i(\tau_0)\Delta\chi^i(\tau_1, \tau_0)\bigg\}.\end{aligned} \tag{4.246b}$$

Equations (4.245) and (4.246) can be used to determine the nonzero components of $\overset{0}{\Omega}{}^A{}_B(\gamma, \bar{\gamma}; \tau_1)$, and then find the values of these components in plane wave spacetimes as a function of initial data $x^i(\tau_0)$, $\xi^i(\tau_0)$, and $\dot{\xi}^i(\tau_0)$. First, the components of $\underset{PP}{\overset{0}{\Omega}}{}^a{}_b(\gamma, \bar{\gamma}; \tau')$ are determined from equation (4.31) to be

$$\underset{PP}{\overset{0}{\Omega}}{}^a{}_b(\gamma, \bar{\gamma}; \tau_1) = \Omega^a{}_b(\gamma, \bar{\gamma}; \tau_1). \tag{4.247}$$

From equations (4.31), (4.245), and (4.246), it is possible to show that the remaining components of $\overset{0}{\Omega}{}^{A}{}_{B}(\gamma, \bar{\gamma}; \tau_1)$ are

$$\underset{JP}{\overset{0}{\Omega}}{}^{uv}{}_{u}(\gamma, \bar{\gamma}; \tau_1) = -\Delta\chi^{v}(\gamma, \bar{\gamma}; \tau_1), \tag{4.248a}$$

$$\underset{JP}{\overset{0}{\Omega}}{}^{ui}{}_{u}(\gamma, \bar{\gamma}; \tau_1) = \underset{JP}{\overset{0}{\Omega}}{}^{vi}{}_{v}(\gamma, \bar{\gamma}; \tau_1) = -\Delta\chi^{i}(\gamma, \bar{\gamma}; \tau_1, \tau_0), \tag{4.248b}$$

$$\underset{JP}{\overset{0}{\Omega}}{}^{vi}{}_{u}(\gamma, \bar{\gamma}; \tau_1) = \Delta\chi^{v}(\gamma, \bar{\gamma}; \tau_1)\Omega^{i}{}_{u}(\gamma, \bar{\gamma}; \tau_1) - \Delta\chi^{i}(\gamma, \bar{\gamma}; \tau_1, \tau_0)\Omega^{v}{}_{u}(\gamma, \bar{\gamma}; \tau_1), \tag{4.248c}$$

$$\underset{JP}{\overset{0}{\Omega}}{}^{vi}{}_{j}(\gamma, \bar{\gamma}; \tau_1) = \Delta\chi^{v}(\gamma, \bar{\gamma}; \tau_1)\delta^{i}{}_{j} - \Delta\chi^{i}(\gamma, \bar{\gamma}; \tau_1, \tau_0)\Omega^{v}{}_{j}(\gamma, \bar{\gamma}; \tau_1), \tag{4.248d}$$

$$\underset{JP}{\overset{0}{\Omega}}{}^{ij}{}_{u}(\gamma, \bar{\gamma}; \tau_1) = 2\Delta\chi^{[i}(\gamma, \bar{\gamma}; \tau_1, \tau_0)\Omega^{j]}{}_{u}(\gamma, \bar{\gamma}; \tau_1), \tag{4.248e}$$

$$\underset{JP}{\overset{0}{\Omega}}{}^{ij}{}_{k}(\gamma, \bar{\gamma}; \tau_1, \tau_0) = 2\Delta\chi^{[i}(\gamma, \bar{\gamma}; \tau_1, \tau_0)\delta^{j]}{}_{k}, \tag{4.248f}$$

$$\underset{JJ}{\overset{0}{\Omega}}{}^{uv}{}_{ui}(\gamma, \bar{\gamma}; \tau_1) = \frac{1}{2}\Omega^{v}{}_{i}(\gamma, \bar{\gamma}; \tau_1), \tag{4.248g}$$

$$\underset{JJ}{\overset{0}{\Omega}}{}^{vi}{}_{uv}(\gamma, \bar{\gamma}; \tau_1) = -\frac{1}{2}\Omega^{i}{}_{u}(\gamma, \bar{\gamma}; \tau_1), \tag{4.248h}$$

$$\underset{JJ}{\overset{0}{\Omega}}{}^{vi}{}_{uj}(\gamma, \bar{\gamma}; \tau_1) = \frac{1}{2}\Big[\delta^{i}{}_{j}\Omega^{v}{}_{u}(\gamma, \bar{\gamma}; \tau_1) - \Omega^{i}{}_{u}(\gamma, \bar{\gamma}; \tau_1)\Omega^{v}{}_{i}(\gamma, \bar{\gamma}; \tau_1)\Big], \tag{4.248i}$$

$$\underset{JJ}{\overset{0}{\Omega}}{}^{vi}{}_{jk}(\gamma, \bar{\gamma}; \tau_1) = -\delta^{i}{}_{[j}\Omega^{v}{}_{k]}(\gamma, \bar{\gamma}; \tau_1), \tag{4.248j}$$

$$\underset{JJ}{\overset{0}{\Omega}}{}^{ij}{}_{uk}(\gamma, \bar{\gamma}; \tau_1) = -\delta^{[i}{}_{k}\Omega^{j]}{}_{u}(\gamma, \bar{\gamma}; \tau_1). \tag{4.248k}$$

These expressions still involve the components of $\Omega^{a}{}_{b}(\gamma, \bar{\gamma}; \tau_1)$ and $\Delta\chi^{a}(\gamma, \bar{\gamma}; \tau_1)$. The former of these already has an expansion in ξ^a and $\dot{\xi}^a$ that was introduced in equation (4.124). We now write a similar expansion for $\Delta\chi^{a}(\gamma, \bar{\gamma}; \tau_1)$:

$$\Delta\chi^{a}(\gamma, \bar{\gamma}; \tau_1) \equiv \sum_{m=1}^{\infty}\sum_{k=0}^{m} {}_{\xi^k\dot{\xi}^{m-k}}\Delta\eta^{a}{}_{b_1\cdots b_k c_1\cdots c_{m-k}}(\tau_1)\xi^{b_1}\cdots\xi^{b_k}\dot{\xi}^{c_1}\cdots\dot{\xi}^{c_{m-k}}. \tag{4.249}$$

The nonzero components of the coefficients in these expansion are given by equations (4.245) and (4.246):

$$ {}_{\xi}\Omega^{v}{}_{ij}(\tau_1) = {}_{\xi}\Omega_{iuj}(\tau_1) = -\partial_{u_1}\underline{K}_{ij}(u_1,u_0), \tag{4.250a}$$

$$ {}_{\dot\xi}\Omega^{v}{}_{ij}(\tau_1) = {}_{\dot\xi}\Omega_{iuj}(\tau_1) = -\frac{1}{\chi}\left\{\partial_{u_1}\left[(u_1-u_0)\underline{H}_{ij}(u_1,u_0)\right]-\delta_{ij}\right\}, \tag{4.250b}$$

$$ {}_{\xi^2}\Omega^{v}{}_{uij}(\tau_1) = \frac{1}{2}\partial_{u_1}\underline{K}_{k(i}(u_1,u_0)\partial_{u_1}\underline{K}^{k}{}_{j)}(u_1,u_0), \tag{4.250c}$$

$$ {}_{\xi\dot\xi}\Omega^{v}{}_{uij}(\tau_1) = \frac{1}{\chi}\partial_{u_1}\underline{K}_{ki}(u_1,u_0)\left\{\partial_{u_1}\left[(u_1-u_0)\underline{H}^{k}{}_{j}(u_1,u_0)\right]-\delta^{k}{}_{j}\right\}, \tag{4.250d}$$

$$ {}_{\dot\xi^2}\Omega^{v}{}_{uij}(\tau_1) = \frac{1}{2\chi^2}\left\{\partial_{u_1}\left[(u_1-u_0)\underline{H}_{k(i}(u_1,u_0)\right]-\delta_{k(i}\right\}\left\{\partial_{u_1}\left[(u_1-u_0)\underline{H}^{k}{}_{j)}(u_1,u_0)\right]-\delta^{k}{}_{j)}\right\}, \tag{4.250e}$$

$$ {}_{\xi}\Delta\eta^{i}{}_{j}(\tau_1,\tau_0) = \delta^{i}{}_{j} - \left[\underline{K}^{i}{}_{j}(u_1,u_0)-(u_1-u_0)\partial_{u_1}\underline{K}^{i}{}_{j}(u_1,u_0)\right], \tag{4.250f}$$

$$ {}_{\dot\xi}\Delta\eta^{i}{}_{j}(\tau_1,\tau_0) = \frac{1}{\chi}(u'-u)^2\partial_{u_1}\underline{H}^{i}{}_{j}(u_1,u_0), \tag{4.250g}$$

$$ {}_{\dot x}[{}_{\xi}\Delta\eta^{v}{}_{i}]_{j}(\tau_1) = \frac{1}{\chi}{}_{\xi}\Delta\eta_{ji}(\tau_1,\tau_0), \tag{4.250h}$$

$$ {}_{\dot x}[{}_{\dot\xi}\Delta\eta^{v}{}_{i}]_{j}(\tau_1) = \frac{1}{\chi}{}_{\dot\xi}\Delta\eta_{ji}(\tau_1,\tau_0), \tag{4.250i}$$

$$ {}_{\xi^2}\Delta\eta^{v}{}_{ij}(\tau_1) = \frac{1}{2}\partial_{u_1}A_{k(i}(u_1,u_0)\left[\underline{K}^{k}{}_{j)}(u_1,u_0)-(u_1-u_0)\partial_{u_1}\underline{K}^{k}{}_{j)}(u_1,u_0)\right], \tag{4.250j}$$

$$ \begin{aligned} {}_{\xi\dot\xi}\Delta\eta^{v}{}_{ij}(\tau_1) = \frac{1}{2\chi}\Big\{ & \left[\underline{K}^{k}{}_{i}(u_1,u_0)-(u_1-u_0)\partial_{u_1}\underline{K}^{k}{}_{i}(u_1,u_0)\right]\left(\partial_{u_1}\left[(u_1-u_0)\underline{H}_{kj}(u_1,u_0)\right]-\delta_{kj}\right) \\ & -(u_1-u_0)^2\partial_{u_1}\underline{K}_{ki}(u_1,u_0)\partial_{u_1}\underline{H}^{k}{}_{j}(u_1,u_0)+\delta_{ij} \\ & -\left[\underline{K}_{ji}(u_1,u_0)-(u_1-u_0)\partial_{u_1}\underline{K}_{ji}(u_1,u_0)\right]\Big\}, \end{aligned} \tag{4.250k}$$

$$ {}_{\dot\xi^2}\Delta\eta^{v}{}_{ij}(\tau_1) = -\frac{1}{\chi^2}(u_1-u_0)^2\left\{\frac{1}{2}\partial_{u_1}\left[(u_1-u_0)\underline{H}_{k(i}(u_1,u_0)\right]-\delta_{k(i}\right\}\partial_{u_1}\underline{H}^{k}{}_{j)}(u_1,u_0). \tag{4.250l}$$

With these components provided, our discussion of the affine transport holonomy is complete.

We turn now to dual Killing transport, where we first discuss the number of independent nonzero components of the holonomy $\overset{1/2}{\Omega}{}^{A}{}_{B}(\gamma,\bar\gamma;\tau_1)$ in plane wave spacetimes. This holonomy, in general spacetimes, has potentially 100 different independent, nonzero components. Because of the five-dimensional space of Killing vector fields in plane wave spacetimes, our final result should have fewer independent components. The easiest way to see this is to note that, for a given Killing vector ξ^a, and for P^a and J^{ab} transported along a curve using dual Killing transport, the quantity

Q defined in equation (4.35) is constant along the curve. In particular, this means that

$$0 = \underset{PP}{\overset{1/2}{\Omega}}{}^c{}_a(\gamma, \bar{\gamma}; \tau_1)\xi_c + \frac{1}{2}\underset{JP}{\overset{1/2}{\Omega}}{}^{cd}{}_a(\gamma, \bar{\gamma}; \tau_1)\nabla_c\xi_d, \tag{4.251a}$$

$$0 = \underset{PJ}{\overset{1/2}{\Omega}}{}^c{}_{ab}(\gamma, \bar{\gamma}; \tau_1)\xi_c + \frac{1}{2}\underset{JJ}{\overset{1/2}{\Omega}}{}^{cd}{}_{ab}(\gamma, \bar{\gamma}; \tau_1)\nabla_c\xi_d \tag{4.251b}$$

[note that this is essentially the same as equation (4.36)] The five Killing vectors for which this equation holds are $\xi_a = \ell_a$ (which satisfies $\nabla_{[a}\xi_{b]} = 0$), and $\xi_a = \Xi_a$, where Ξ_a is given by equation (4.212), and thus satisfy

$$\nabla_{[a}\xi_{b]} = 2\dot{\Xi}_i(u_0)\ell_{[a}(\mathrm{d}x^i)_{b]}. \tag{4.252}$$

Therefore, equations (4.251) imply that

$$\underset{PP}{\overset{1/2}{\Omega}}{}^u{}_\mu(\gamma, \bar{\gamma}; \tau_1) = 0, \qquad \underset{PJ}{\overset{1/2}{\Omega}}{}^u{}_{\mu\nu}(\gamma, \bar{\gamma}; \tau_1) = 0, \tag{4.253a}$$

$$\underset{PP}{\overset{1/2}{\Omega}}{}^i{}_\mu(\gamma, \bar{\gamma}; \tau_1) = 0, \qquad \underset{PJ}{\overset{1/2}{\Omega}}{}^i{}_{\mu\nu}(\gamma, \bar{\gamma}; \tau_1) = 0, \tag{4.253b}$$

$$\underset{JP}{\overset{1/2}{\Omega}}{}^{ui}{}_\mu(\gamma, \bar{\gamma}; \tau_1) = 0, \qquad \underset{JJ}{\overset{1/2}{\Omega}}{}^{ui}{}_{\mu\nu}(\gamma, \bar{\gamma}; \tau_1) = 0. \tag{4.253c}$$

Here, equation (4.253a) corresponds to $\xi^a = \ell^a$, while equations (4.253b) and (4.253c) correspond to $\xi^a = \Xi^a$, and are the constraints due to varying over the initial data $\Xi^i(u_0)$ and $\dot{\Xi}^i(u_0)$ in equation (4.214), respectively. This reduces the number of possible independent components to 50.

We turn now to the actual computation of this holonomy. Much like the affine transport holonomy can be written in terms of the holonomy of the metric-compatible connection, the holonomy of dual Killing transport can be written in terms of the holonomy of affine transport. To show this, we take advantage of the fact that the beginning and end of the loop are in the flat regions of spacetime. In these regions, there is no difference between affine transport and dual Killing transport (the value of $\varkappa$ is irrelevant, as the Riemann tensor vanishes). Therefore, we can compute the holonomy by using different values of $\varkappa$ along different segments of the loop. This yields

$$\begin{aligned}\overset{1/2}{\Omega}{}^A{}_B(\gamma, \bar{\gamma}; \tau_1) = \Big\{ &\left[\delta^A{}_C + \hat{\Delta}^A{}_C(\gamma, \bar{\gamma}; \tau_1)\right] \overset{0}{\Omega}{}^C{}_D(\gamma, \bar{\gamma}; \tau_1) \\ &+ \hat{\Delta}^A{}_D(\gamma, \bar{\gamma}; \tau_1) - \hat{\Delta}^A{}_D(\gamma, \gamma; \tau_1)\Big\} \left[\delta^D{}_B + \Delta^D{}_B(\gamma; \tau_1)\right],\end{aligned} \tag{4.254}$$

where

$$\Delta^A{}_B(\gamma; \tau_1) \equiv {}_\gamma\overset{0}{g}{}^A{}_{A'}\,{}_\gamma\overset{1/2}{g}{}^{A'}{}_B - \delta^A{}_B, \tag{4.255a}$$

$$\hat{\Delta}^A{}_B(\gamma, \bar{\gamma}; \tau_1) \equiv \overset{0}{g}{}^A{}_{\bar{A}}\,{}_{\bar{\gamma}}\overset{1/2}{g}{}^{\bar{A}}{}_{\bar{A}'}\,{}_\gamma\overset{0}{g}{}^{\bar{A}'}{}_{\bar{B}}\overset{0}{g}{}^{\bar{B}}{}_B - \delta^A{}_B. \tag{4.255b}$$

Note that in equation (4.254), both $\hat{\Delta}^A{}_B(\gamma,\bar{\gamma};\tau_1)$ and $\hat{\Delta}^A{}_B(\gamma,\gamma;\tau_1)$ appear. The latter is defined by equation (4.255b), but with $\bar{\gamma}=\gamma$, which implies that $\bar{x}=x$ and $\bar{x}'=x'$; equivalently, $\hat{\Delta}^A{}_B(\gamma,\gamma;\tau_1)$ is the same as $\Delta^A{}_B(\gamma;\tau_1)$, but with the order of the parallel propagators reversed.

At this point, we now give expressions for the various terms that occur in equation (4.254). The key point to take away is that all components of the tensors that occur can be written solely in terms of the transverse Jacobi propagators $\underline{K}^i{}_j(u',u)$ and $\underline{H}^i{}_j(u',u)$ and their derivatives. The components of $\Delta^A{}_B(\gamma;\tau_1)$ and $\hat{\Delta}^A{}_B(\gamma,\gamma;\tau_1)$ are given by a lengthy calculation starting with their definitions in equation (4.255), expressions for the parallel propagator for dual Killing transport in equations (4.164) and (4.165), and equations (4.225), along with the following formulae that hold in plane wave spacetimes:

$$
{}_{\xi}\Omega_{ab}{}^c(\gamma;\tau_1) = \frac{2}{\chi}\ell_{[a\,\gamma}g_{b]}{}^{a'}\frac{\mathrm{D}\,{}_{\gamma}K_{a'}{}^c}{\mathrm{d}\tau_1}, \tag{4.256a}
$$

$$
{}_{\dot{\xi}}\Omega_{ab}{}^c(\gamma;\tau_1) = \frac{2}{\chi}\ell_{[a\,\gamma}g_{b]}{}^{a'}\frac{\mathrm{D}\left[(\tau'-\tau)({}_{\gamma}H_{a'}{}^c - {}_{\gamma}g_{a'}{}^c)\right]}{\mathrm{d}\tau_1}. \tag{4.256b}
$$

At the end of this computation, one finds that

$$
\underset{PP}{\Delta}{}^i{}_j(\gamma;\tau_1,\tau_0) = \chi\,{}_{\dot{x}}[\underset{PP}{\Delta}{}^v{}_j]^i(\gamma;\tau_1) = -\chi\,{}_{\dot{x}}[\underset{PP}{\Delta}{}^i{}_u]_j(\gamma;\tau_1) = \underline{K}_j{}^i(u_0,u_1) - \delta_j{}^i, \tag{4.257a}
$$

$$
{}_{\dot{x}^2}[\underset{PP}{\Delta}{}^v{}_u]_{ij}(\gamma;\tau_1) = -\frac{1}{\chi^2}\underset{PP}{\Delta}{}_{(ij)}(\gamma;\tau_1,\tau_0), \tag{4.257b}
$$

$$
\begin{aligned}
\underset{PJ}{\Delta}{}^i{}_{uj}(\gamma;\tau_1) &= \chi\,{}_{\dot{x}}[\underset{PJ}{\Delta}{}^v{}_{uj}]^i(\gamma;\tau_1) \\
&= -\frac{1}{2}\left[\underline{K}_k{}^i(u_0,u_1)\partial_{u_1}\underline{K}_j{}^k(u_1,u_0) + \partial_{u_0}\underline{K}_k{}^i\partial_{u_1}\left\{(u_1-u_0)[\underline{H}_j{}^k(u_1,u_0)-\delta_j{}^k]\right\}\right],
\end{aligned} \tag{4.257c}
$$

$$\begin{aligned}\underset{JP}{\Delta}{}^{ui}{}_{j}(\gamma;\tau_1) &= \chi\, \dot{x}[\underset{JP}{\Delta}{}^{uv}{}_{j}]^{i}(\gamma;\tau_1) = -\chi\, \dot{x}[\underset{JP}{\Delta}{}^{ui}{}_{u}]_{j}(\gamma;\tau_1) = 2\chi^2[\underset{JP}{\Delta}{}^{vi}{}_{j}](\gamma;\tau_1)\\ &= -2\chi^3\, \dot{x}[\underset{JP}{\Delta}{}^{vi}{}_{u}]_{j}(\gamma;\tau_1) = (u_1-u_0)\left[\underline{K}_j{}^{i}(u_0,u_1) - \underline{H}_j{}^{i}(u_0,u_1)\right], \qquad (4.257\text{d})\end{aligned}$$

$$\dot{x}^2[\underset{JP}{\Delta}{}^{uv}{}_{u}]_{ij}(\gamma;\tau_1) = -\frac{1}{\chi^2}\underset{JP}{\Delta}{}^{u}{}_{(ij)}(\gamma;\tau_1), \qquad (4.257\text{e})$$

$$\dot{x}[\underset{JP}{\Delta}{}^{ij}{}_{k}]_{l}(\gamma;\tau_1) = \frac{2}{\chi}\delta^{[i}{}_{l}\underset{JP}{\Delta}{}^{|u|j]}{}_{k}(\gamma;\tau_1), \qquad (4.257\text{f})$$

$$\dot{x}^2[\underset{JP}{\Delta}{}^{ij}{}_{u}]_{kl}(\gamma;\tau_1) = -\frac{1}{\chi}\, \dot{x}[\underset{JP}{\Delta}{}^{ij}{}_{(k}]_{l)}(\gamma;\tau_1), \qquad (4.257\text{g})$$

$$\dot{x}^2[\underset{JP}{\Delta}{}^{vi}{}_{j}]_{kl}(\gamma;\tau_1) = \delta_{kl}[\underset{JP}{\Delta}{}^{vi}{}_{j}](\gamma;\tau_1) - 2\delta^{i}{}_{(k}[\underset{JP}{\Delta}{}^{v}{}_{l)j}](\gamma;\tau_1), \qquad (4.257\text{h})$$

$$\dot{x}^3[\underset{JP}{\Delta}{}^{vi}{}_{u}]_{jkl}(\gamma;\tau_1) = -\frac{1}{\chi}\, \dot{x}^2[\underset{JP}{\Delta}{}^{vi}{}_{(j}]_{kl)}(\gamma;\tau_1), \qquad (4.257\text{i})$$

$$\begin{aligned}\underset{JJ}{\Delta}{}^{ui}{}_{uj}(\gamma;\tau_1) &= \chi\, \dot{x}[\underset{JJ}{\Delta}{}^{uv}{}_{uj}]^{i}(\gamma;\tau_1) = 2\chi^2[\underset{JJ}{\Delta}{}^{vi}{}_{uj}](\gamma;\tau_1)\\ &= \frac{1}{2}(u_1-u_0)\left[\underline{H}_k{}^{i}(u_0,u_1) - \underline{K}_k{}^{i}(u_0,u_1)\right]\partial_{u_1}\underline{K}_j{}^{k}(u_1,u_0)\\ &\quad + \frac{1}{2}\left[(u_1-u_0)\partial_{u_1}\underline{K}^{i}{}_{k}(u_1,u_0) - \underline{K}^{i}{}_{k}(u_1,u_0)\right]\\ &\qquad \times \partial_{u_1}\left\{(u_1-u_0)\left[\underline{H}_j{}^{k}(u_1,u_0) - \delta_j{}^{k}\right]\right\}, \qquad (4.257\text{j})\end{aligned}$$

$$\dot{\xi}[\underset{JJ}{\Delta}{}^{ij}{}_{uk}]_{l}(\gamma;\tau_1) = \frac{2}{\chi}\delta^{[i}{}_{l}\underset{JJ}{\Delta}{}^{|u|j]}{}_{uk}(\gamma;\tau_1), \qquad (4.257\text{k})$$

$$\dot{x}^2[\underset{JJ}{\Delta}{}^{vi}{}_{uj}]_{kl}(\gamma;\tau_1) = \delta_{kl}[\underset{JJ}{\Delta}{}^{vi}{}_{uj}](\gamma;\tau_1) - 2\delta^{i}{}_{(k}[\underset{JJ}{\Delta}{}^{v}{}_{l)uj}](\gamma;\tau_1), \qquad (4.257\text{l})$$

and

$$\begin{aligned}\underset{PP}{\hat{\Delta}}{}^{i}{}_{j}(\gamma,\gamma;\tau_1,\tau_0) &= \chi\, \dot{x}[\underset{PP}{\hat{\Delta}}{}^{v}{}_{j}]^{i}(\gamma,\gamma;\tau_1) = -\chi\, \dot{x}[\underset{PP}{\hat{\Delta}}{}^{i}{}_{u}]_{j}(\gamma,\gamma;\tau_1)\\ &= \underline{K}_j{}^{i}(u_1,u_0) - \delta_j{}^{i} - (u_1-u_0)\partial_{u_1}\underline{K}_j{}^{i}(u_1,u_0), \qquad (4.258\text{a})\end{aligned}$$

$$\dot{x}^2[\underset{PP}{\hat{\Delta}}{}^{v}{}_{u}]_{ij}(\gamma,\gamma;\tau_1) = -\frac{1}{\chi^2}\underset{PP}{\hat{\Delta}}{}_{(ij)}(\gamma,\gamma;\tau_1,\tau_0), \qquad (4.258\text{b})$$

$$\underset{PJ}{\hat{\Delta}}{}^{i}{}_{uj}(\gamma,\gamma;\tau_1) = \chi\, \dot{x}[\underset{PJ}{\hat{\Delta}}{}^{v}{}_{uj}]^{i}(\gamma,\gamma;\tau_1) = \frac{1}{2}\partial_{u_1}\underline{K}_j{}^{i}(u_1,u_0), \qquad (4.258\text{c})$$

$$\underset{JP}{\hat{\Delta}}{}^{ui}{}_{j}(\gamma,\gamma;\tau_1) = \chi\,{}_{\dot{x}}[\underset{JP}{\hat{\Delta}}{}^{uv}{}_{j}]^{i}(\gamma,\gamma;\tau_1) = -\chi\,{}_{\dot{x}}[\underset{JP}{\hat{\Delta}}{}^{ui}{}_{u}]_{j}(\gamma,\gamma;\tau_1) = 2\chi^2[\underset{JP}{\hat{\Delta}}{}^{vi}{}_{j}](\gamma,\gamma;\tau_1)$$

$$= -2\chi^3[\underset{JP}{\hat{\Delta}}{}^{vi}{}_{u}]_{j}(\gamma,\gamma;\tau_1) = -(u_1-u_0)^2\partial_{u_1}\underline{H}_{j}{}^{i}(u_1,u_0), \tag{4.258d}$$

$${}_{\dot{x}^2}[\underset{JP}{\hat{\Delta}}{}^{uv}{}_{u}]_{ij}(\gamma,\gamma;\tau_1) = -\frac{1}{\chi^2}\underset{JP}{\hat{\Delta}}{}^{u}{}_{(ij)}(\gamma,\gamma;\tau_1), \tag{4.258e}$$

$${}_{\dot{x}}[\underset{JP}{\hat{\Delta}}{}^{ij}{}_{k}]_{l}(\gamma,\gamma;\tau_1,\tau_0) = \frac{2}{\chi}\delta^{[i}{}_{l}\underset{JP}{\hat{\Delta}}{}^{|u|j]}{}_{k}(\gamma,\gamma;\tau_1), \tag{4.258f}$$

$${}_{\dot{x}^2}[\underset{JP}{\hat{\Delta}}{}^{ij}{}_{u}]_{kl}(\gamma,\gamma;\tau_1) = -\frac{1}{\chi}\,{}_{\dot{x}}[\underset{JP}{\hat{\Delta}}{}^{ij}{}_{(k}]_{l)}(\gamma,\gamma;\tau_1,\tau_0), \tag{4.258g}$$

$${}_{\dot{x}^2}[\underset{JP}{\hat{\Delta}}{}^{vi}{}_{j}]_{kl}(\gamma,\gamma;\tau_1) = \delta_{kl}[\underset{JP}{\hat{\Delta}}{}^{vi}{}_{j}](\gamma,\gamma;\tau_1) - 2\delta^{i}{}_{(k}[\underset{JP}{\hat{\Delta}}{}^{v}{}_{l)j}](\gamma,\gamma;\tau_1), \tag{4.258h}$$

$${}_{\dot{x}^3}[\underset{JP}{\hat{\Delta}}{}^{vi}{}_{u}]_{jkl}(\gamma,\gamma;\tau_1) = -\frac{1}{\chi}\,{}_{\dot{x}^2}[\underset{JP}{\hat{\Delta}}{}^{vi}{}_{(j}]_{kl)}(\gamma,\gamma;\tau_1), \tag{4.258i}$$

$$\underset{JJ}{\hat{\Delta}}{}^{ui}{}_{uj}(\gamma,\gamma;\tau_1) = \chi\,{}_{\dot{x}}[\underset{JJ}{\hat{\Delta}}{}^{uv}{}_{uj}]^{i}(\gamma,\gamma;\tau_1) = 2\chi^2[\underset{JJ}{\hat{\Delta}}{}^{vi}{}_{uj}](\gamma,\gamma;\tau_1)$$

$$= \frac{1}{2}\partial_{u_1}\left\{(u_1-u_0)\left[\underline{H}_{j}{}^{i}(u_1,u_0) - \delta_{j}{}^{i}\right]\right\}, \tag{4.258j}$$

$${}_{\dot{x}}[\underset{JJ}{\hat{\Delta}}{}^{ij}{}_{uk}]_{l}(\gamma,\gamma;\tau_1) = \frac{2}{\chi}\delta^{[i}{}_{l}\underset{JJ}{\hat{\Delta}}{}^{|u|j]}{}_{uk}(\gamma,\gamma;\tau_1), \tag{4.258k}$$

$${}_{\dot{x}^2}[\underset{JJ}{\hat{\Delta}}{}^{vi}{}_{uj}]_{kl}(\gamma,\gamma;\tau_1) = \delta_{kl}[\underset{JJ}{\hat{\Delta}}{}^{vi}{}_{uj}](\gamma,\gamma;\tau_1) - 2\delta^{i}{}_{(k}[\underset{JJ}{\hat{\Delta}}{}^{v}{}_{l)uj}](\gamma,\gamma;\tau_1). \tag{4.258l}$$

There are now two additional pieces of notation that we must introduce in order to finish this computation. First, we expand $\hat{\Delta}^{A}{}_{B}(\gamma,\bar{\gamma};\tau_1)$ in powers of the separation:

$$\hat{\Delta}^{A}{}_{B}(\gamma,\bar{\gamma};\tau_1) \equiv \sum_{n=0}^{\infty} {}_{\xi^n}\Theta^{A}{}_{Bc_1\cdots c_n}(\gamma,\bar{\gamma};\tau_1)\xi^{c_1}\cdots\xi^{c_n}. \tag{4.259}$$

Next, it happens that in plane wave spacetimes, the components of the coefficients in this expansion depend on the sum $\dot{\bar{x}}^i(\tau_0) = \dot{x}^i(\tau_0) + \dot{\xi}^i(\tau_0)$, and not independently on either $\dot{x}^i(\tau_0)$ or $\dot{\xi}^i(\tau_0)$. As such, we write [in analogy to equation (4.235) above] such quantities in terms of coefficients of the following expansion:

$$Q^{\cdots}{}_{\cdots} = \sum_{k=0}^{n} {}_{\dot{\bar{x}}^k}[Q^{\cdots}{}_{\cdots}]_{i_1\cdots i_k}\dot{\bar{x}}^{i_1}(\tau_0)\cdots\dot{\bar{x}}^{i_k}(\tau_0). \tag{4.260}$$

In this notation, we can show that the components of $\Theta^{A}{}_{B}(\gamma,\bar{\gamma};\tau_1)$ and $\hat{\Delta}^{A}{}_{B}(\gamma,\gamma;\tau_1)$ are related:

$${}_{\dot{\bar{x}}^k}[\Theta^{\Gamma}{}_{\Delta}](\gamma,\bar{\gamma};\tau_1) = {}_{\dot{x}^k}[\hat{\Delta}^{\Gamma}{}_{\Delta}](\gamma,\gamma;\tau_1), \tag{4.261}$$

using Γ and Δ for Brinkmann coordinate indices on the linear and angular momentum bundle. Using this notation, the final nonzero components that are needed are also given by the calculation mentioned above:

$$ {}_{\xi}\underset{PP}{\Theta}{}^{i}{}_{uj}(\gamma,\bar\gamma;\tau_1) = \chi\,{}_{\dot{x}}[{}_{\xi}\underset{PP}{\Theta}{}^{v}{}_{uj}]^{i}(\gamma,\bar\gamma;\tau_1) = \partial_{u'}\underline{K}_j{}^{i}(u_1,u_0), \tag{4.262a} $$

$$ {}_{\xi}\underset{JP}{\Theta}{}^{vi}{}_{jv}(\gamma,\bar\gamma;\tau_1) = -\chi\,{}_{\dot{x}}[{}_{\xi}\underset{JP}{\Theta}{}^{vi}{}_{uv}]_j(\gamma,\bar\gamma;\tau_1) = \underline{K}_j{}^{i}(u_1,u_0) - \delta_j{}^{i} - (u_1-u_0)\partial_{u_1}\underline{K}_j{}^{i}(u_1,u_0), \tag{4.262b} $$

$$ {}_{\dot{x}}[{}_{\xi}\underset{JP}{\Theta}{}^{iv}{}_{jk}]_l(\gamma,\bar\gamma;\tau_1) = \frac{1}{\chi}\delta^{i}{}_{k}\,{}_{\xi}\underset{JP}{\Theta}{}^{v}{}_{ljv}(\gamma,\bar\gamma;\tau_1), \tag{4.262c} $$

$$ {}_{\xi}\underset{JP}{\Theta}{}^{ij}{}_{kl}(\gamma,\bar\gamma;\tau_1,\tau_0) = 2\chi\,{}_{\dot{x}}[{}_{\xi}\underset{JP}{\Theta}{}^{[i|v|}{}_{kl}]^{j]}(\gamma,\bar\gamma;\tau_1), \tag{4.262d} $$

$$ \begin{aligned} {}_{\xi}\underset{JP}{\Theta}{}^{ui}{}_{uj}(\gamma,\bar\gamma;\tau_1) &= \chi\,{}_{\dot{x}}[{}_{\xi}\underset{JP}{\Theta}{}^{uv}{}_{uj}](\gamma,\bar\gamma;\tau_1) = 2\chi^2[{}_{\xi}\underset{JP}{\Theta}{}^{vi}{}_{uj}](\gamma,\bar\gamma;\tau_1) \\ &= \partial_{u_1}\left\{(u_1-u_0)\left[\underline{H}_j{}^{i}(u_1,u_0)-\delta_j{}^{i}\right]\right\}, \end{aligned} \tag{4.262e} $$

$$ {}_{\dot{x}}[{}_{\xi}\underset{JP}{\Theta}{}^{ij}{}_{uk}]_l(\gamma,\bar\gamma;\tau_1) = -\frac{1}{\chi}\,{}_{\xi}\underset{JP}{\Theta}{}^{ij}{}_{lk}(\gamma,\bar\gamma;\tau_1,\tau_0) + \frac{2}{\chi}\delta^{[i}{}_{l}\,{}_{\xi}\underset{JP}{\Theta}{}^{|u|j]}{}_{uk}(\gamma,\bar\gamma;\tau_1), \tag{4.262f} $$

$$ {}_{\dot{x}^2}[{}_{\xi}\underset{JP}{\Theta}{}^{vi}{}_{uj}]_{kl}(\gamma,\bar\gamma;\tau_1) = -\frac{1}{\chi}\,{}_{\dot{x}}[{}_{\xi}\underset{JP}{\Theta}{}^{iv}{}_{(k|j|}]_{l)}(\gamma,\bar\gamma;\tau_1) + \delta_{kl}[{}_{\xi}\underset{JP}{\Theta}{}^{vi}{}_{uj}](\gamma,\bar\gamma;\tau_1) - 2\delta^{i}{}_{(k}[{}_{\xi}\underset{JP}{\Theta}{}^{v}{}_{l)uj}](\gamma,\bar\gamma;\tau_1), \tag{4.262g} $$

$$ {}_{\xi}\underset{JJ}{\Theta}{}^{vi}{}_{ujv}(\gamma,\bar\gamma;\tau_1) = \frac{1}{2}\partial_{u_1}\underline{K}_j{}^{i}(u_1,u_0), \tag{4.262h} $$

$$ {}_{\dot{x}}[{}_{\xi}\underset{JJ}{\Theta}{}^{iv}{}_{ujk}]_l(\gamma,\bar\gamma;\tau_1) = \frac{1}{\chi}\delta^{i}{}_{k}\underset{JJ}{\Theta}{}^{v}{}_{lujv}(\gamma,\bar\gamma;\tau_1), \tag{4.262i} $$

$$ {}_{\xi}\underset{JJ}{\Theta}{}^{ij}{}_{ukl}(\gamma,\bar\gamma;\tau_1) = 2\chi\,{}_{\dot{x}}[{}_{\xi}\underset{JJ}{\Theta}{}^{[i|v|}{}_{ukl}]^{j]}(\gamma,\bar\gamma;\tau_1). \tag{4.262j} $$

For our final result, we first define

$$ \Xi^{A}{}_{B}(\gamma,\bar\gamma;\tau_1) \equiv \overset{0}{\Omega}{}^{A}{}_{B}(\gamma,\bar\gamma;\tau_1) + \hat\Delta^{A}{}_{B}(\gamma,\bar\gamma;\tau_1) - \hat\Delta^{A}{}_{B}(\gamma,\gamma;\tau_1); \tag{4.263} $$

then, equation (4.254) implies that

$$ \begin{aligned} \underset{PP}{\overset{1/2}{\Omega}}{}^{v}{}_{u}(\gamma,\bar\gamma;\tau_1) = {}& \underset{PP}{\Xi}{}^{v}{}_{u}(\gamma,\bar\gamma;\tau_1) + \underset{PP}{\Xi}{}^{v}{}_{i}(\gamma,\bar\gamma;\tau_1)\underset{PP}{\Delta}{}^{i}{}_{u}(\gamma;\tau_1) + \underset{PP}{\hat\Delta}{}^{v}{}_{i}(\gamma,\bar\gamma;\tau_1)\underset{PP}{\overset{0}{\Omega}}{}^{i}{}_{u}(\gamma,\bar\gamma;\tau_1) \\ &+ 2\left[\underset{PJ}{\hat\Delta}{}^{v}{}_{ui}(\gamma,\bar\gamma;\tau_1) - \underset{PJ}{\hat\Delta}{}^{v}{}_{ui}(\gamma,\gamma;\tau_1)\right]\underset{JP}{\Delta}{}^{ui}{}_{u}(\gamma;\tau_1) \\ &+ 2\underset{PJ}{\hat\Delta}{}^{v}{}_{ui}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\overset{0}{\Omega}}{}^{ui}{}_{u}(\gamma,\bar\gamma;\tau_1), \end{aligned} \tag{4.264a} $$

$$ \begin{aligned} \underset{PP}{\overset{1/2}{\Omega}}{}^{v}{}_{i}(\gamma,\bar\gamma;\tau_1) = {}& \underset{PP}{\Xi}{}^{v}{}_{j}(\gamma,\bar\gamma;\tau_1)\left[\delta^{j}{}_{i} + \underset{PP}{\Delta}{}^{j}{}_{i}(\gamma;\tau_1)\right] \\ &+ 2\left[\underset{PJ}{\hat\Delta}{}^{v}{}_{uj}(\gamma,\bar\gamma;\tau_1) - \underset{PJ}{\hat\Delta}{}^{v}{}_{uj}(\gamma,\gamma;\tau_1)\right]\underset{JP}{\Delta}{}^{uj}{}_{i}(\gamma;\tau_1), \end{aligned} \tag{4.264b} $$

$$\begin{aligned}\overset{1/2}{\underset{PJ}{\Omega}}{}^{v}{}_{ui}(\gamma,\bar\gamma;\tau_1) = {} & \underset{PP}{\Xi}{}^{v}{}_{j}(\gamma,\bar\gamma;\tau_1)\underset{PP}{\Delta}{}^{j}{}_{ui}(\gamma;\tau_1) + 2\left[\underset{PJ}{\hat\Delta}{}^{v}{}_{uj}(\gamma,\bar\gamma;\tau_1) - \underset{PJ}{\hat\Delta}{}^{v}{}_{uj}(\gamma,\gamma;\tau_1)\right]\underset{JJ}{\Delta}{}^{uj}{}_{ui}(\gamma;\tau_1) \\ & + \underset{PJ}{\hat\Delta}{}^{v}{}_{ui}(\gamma,\bar\gamma;\tau_1) - \underset{PJ}{\hat\Delta}{}^{v}{}_{ui}(\gamma,\gamma;\tau_1),\end{aligned} \tag{4.264c}$$

$$\begin{aligned}\overset{1/2}{\underset{JP}{\Omega}}{}^{uv}{}_{u}(\gamma,\bar\gamma;\tau_1) = {} & \underset{JP}{\Xi}{}^{uv}{}_{u}(\gamma,\bar\gamma;\tau_1) + 2\underset{JJ}{\Xi}{}^{uv}{}_{ui}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{ui}{}_{u}(\gamma;\tau_1) \\ & + \left[\underset{JP}{\hat\Delta}{}^{uv}{}_{i}(\gamma,\bar\gamma;\tau_1) - \underset{JP}{\hat\Delta}{}^{uv}{}_{i}(\gamma,\gamma;\tau_1)\right]\underset{PP}{\Delta}{}^{i}{}_{u}(\gamma;\tau_1) \\ & + \underset{JP}{\hat\Delta}{}^{uv}{}_{i}(\gamma,\bar\gamma;\tau_1)\overset{0}{\underset{PP}{\Omega}}{}^{i}{}_{u}(\gamma,\bar\gamma;\tau_1) + 2\underset{JJ}{\hat\Delta}{}^{uv}{}_{ui}(\gamma,\bar\gamma;\tau_1)\overset{0}{\underset{JP}{\Omega}}{}^{ui}{}_{u}(\gamma,\bar\gamma;\tau_1),\end{aligned} \tag{4.264d}$$

$$\begin{aligned}\overset{1/2}{\underset{JP}{\Omega}}{}^{uv}{}_{i}(\gamma,\bar\gamma;\tau_1) = {} & 2\underset{JJ}{\Xi}{}^{uv}{}_{ui}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{uj}{}_{i}(\gamma;\tau_1) \\ & + \left[\underset{JP}{\hat\Delta}{}^{uv}{}_{j}(\gamma,\bar\gamma;\tau_1) - \underset{JP}{\hat\Delta}{}^{uv}{}_{j}(\gamma,\gamma;\tau_1)\right]\left[\delta^{j}{}_{i} + \underset{PP}{\Delta}{}^{j}{}_{i}(\gamma;\tau_1)\right],\end{aligned} \tag{4.264e}$$

$$\begin{aligned}\overset{1/2}{\underset{JP}{\Omega}}{}^{vi}{}_{u}(\gamma,\bar\gamma;\tau_1) = {} & \overset{0}{\underset{JP}{\Omega}}{}^{vi}{}_{v}(\gamma,\bar\gamma;\tau_1)\underset{PP}{\Delta}{}^{v}{}_{u}(\gamma;\tau_1) + \underset{JP}{\Xi}{}^{vi}{}_{u}(\gamma,\bar\gamma;\tau_1) + \underset{JP}{\Xi}{}^{vi}{}_{j}(\gamma,\bar\gamma;\tau_1)\underset{PP}{\Delta}{}^{i}{}_{u}(\gamma;\tau_1) \\ & + 2\underset{JJ}{\Xi}{}^{vi}{}_{uj}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{uj}{}_{u}(\gamma;\tau_1) + 2\overset{0}{\underset{JJ}{\Omega}}{}^{vi}{}_{uv}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{uv}{}_{u}(\gamma,\bar\gamma;\tau_1) \\ & + \overset{0}{\underset{JJ}{\Omega}}{}^{vi}{}_{jk}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{jk}{}_{u}(\gamma;\tau_1) + \underset{JP}{\hat\Delta}{}^{vi}{}_{j}(\gamma,\bar\gamma;\tau_1)\overset{0}{\underset{PP}{\Omega}}{}^{i}{}_{u}(\gamma,\bar\gamma;\tau_1) \\ & + 2\underset{JJ}{\hat\Delta}{}^{vi}{}_{uj}(\gamma,\bar\gamma;\tau_1)\overset{0}{\underset{JP}{\Omega}}{}^{uj}{}_{u}(\gamma,\bar\gamma;\tau_1),\end{aligned} \tag{4.264f}$$

$$\overset{1/2}{\underset{JP}{\Omega}}{}^{vi}{}_{v}(\gamma,\bar\gamma;\tau_1) = \overset{0}{\underset{JP}{\Omega}}{}^{vi}{}_{v}(\gamma,\bar\gamma;\tau_1), \tag{4.264g}$$

$$\begin{aligned}\overset{1/2}{\underset{JP}{\Omega}}{}^{vi}{}_{j}(\gamma,\bar\gamma;\tau_1) = {} & \underset{JP}{\Xi}{}^{vi}{}_{k}(\gamma,\bar\gamma;\tau_1)\left[\delta^{k}{}_{j} + \underset{PP}{\Delta}{}^{k}{}_{j}(\gamma;\tau_1,\tau_0)\right] + 2\underset{JJ}{\Xi}{}^{vi}{}_{uk}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{uk}{}_{j}(\gamma;\tau_1) \\ & + 2\overset{0}{\underset{JJ}{\Omega}}{}^{vi}{}_{uv}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{uv}{}_{j}(\gamma;\tau_1) + \overset{0}{\underset{JP}{\Omega}}{}^{vi}{}_{v}(\gamma,\bar\gamma;\tau_1)\underset{PP}{\Delta}{}^{v}{}_{j}(\gamma;\tau_1) \\ & + \overset{0}{\underset{JJ}{\Omega}}{}^{vi}{}_{kl}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{kl}{}_{j}(\gamma;\tau_1,\tau_0),\end{aligned} \tag{4.264h}$$

$$\begin{aligned}\overset{1/2}{\underset{JP}{\Omega}}{}^{ij}{}_{u}(\gamma,\bar\gamma;\tau_1) = {} & \underset{JP}{\Xi}{}^{ij}{}_{u}(\gamma,\bar\gamma;\tau_1) + \underset{JP}{\Xi}{}^{ij}{}_{k}(\gamma,\bar\gamma;\tau_1,\tau_0)\underset{PP}{\Delta}{}^{k}{}_{u}(\gamma;\tau_1) + 2\underset{JJ}{\Xi}{}^{ij}{}_{uk}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{uk}{}_{u}(\gamma;\tau_1) \\ & + \underset{JP}{\hat\Delta}{}^{ij}{}_{k}(\gamma,\bar\gamma;\tau_1,\tau_0)\overset{0}{\underset{PP}{\Omega}}{}^{k}{}_{u}(\gamma,\bar\gamma;\tau_1) + 2\underset{JJ}{\hat\Delta}{}^{ij}{}_{uk}(\gamma,\bar\gamma;\tau_1)\overset{0}{\underset{JP}{\Omega}}{}^{uk}{}_{u}(\gamma,\bar\gamma;\tau_1),\end{aligned} \tag{4.264i}$$

$$\overset{1/2}{\underset{JP}{\Omega}}{}^{ij}{}_{k}(\gamma,\bar\gamma;\tau_1,\tau_0) = \underset{JP}{\Xi}{}^{ij}{}_{l}(\gamma,\bar\gamma;\tau_1,\tau_0)\left[\delta^{l}{}_{k} + \underset{PP}{\Delta}{}^{l}{}_{k}(\gamma;\tau_1,\tau_0)\right] + 2\underset{JJ}{\Xi}{}^{ij}{}_{ul}(\gamma,\bar\gamma;\tau_1)\underset{JP}{\Delta}{}^{ul}{}_{k}(\gamma;\tau_1), \tag{4.264j}$$

$$\underset{JJ}{\overset{1/2}{\Omega}}{}^{uv}{}_{ui}(\gamma,\bar{\gamma};\tau_1) = \underset{JJ}{\Xi}{}^{uv}{}_{uj}(\gamma,\bar{\gamma};\tau_1)\left[\delta^j{}_i + 2\underset{JJ}{\Delta}{}^{uj}{}_{ui}(\gamma;\tau_1)\right] + \left[\underset{JP}{\hat{\Delta}}{}^{uv}{}_j(\gamma,\bar{\gamma};\tau_1) - \underset{JP}{\hat{\Delta}}{}^{uv}{}_j(\gamma,\gamma;\tau_1)\right]\underset{PJ}{\Delta}{}^j{}_{ui}(\gamma;\tau_1), \tag{4.264k}$$

$$\underset{JJ}{\overset{1/2}{\Omega}}{}^{vi}{}_{uv}(\gamma,\bar{\gamma};\tau_1) = \underset{JJ}{\overset{0}{\Omega}}{}^{vi}{}_{uv}(\gamma,\bar{\gamma};\tau_1), \tag{4.264l}$$

$$\begin{aligned}\underset{JJ}{\overset{1/2}{\Omega}}{}^{vi}{}_{uj}(\gamma,\bar{\gamma};\tau_1) = {}& \underset{JP}{\Xi}{}^{vi}{}_k(\gamma,\bar{\gamma};\tau_1)\underset{PJ}{\Delta}{}^k{}_{uj}(\gamma;\tau_1) + \underset{JJ}{\Xi}{}^{vi}{}_{uk}(\gamma,\bar{\gamma};\tau_1)\left[\delta^k{}_j + 2\underset{JJ}{\Delta}{}^{uk}{}_{uj}(\gamma;\tau_1)\right]\\ &+ \underset{JP}{\overset{0}{\Omega}}{}^{vi}{}_v(\gamma,\bar{\gamma};\tau_1)\underset{PJ}{\Delta}{}^v{}_{uj}(\gamma;\tau_1) + \underset{JJ}{\overset{0}{\Omega}}{}^{vi}{}_{kl}(\gamma,\bar{\gamma};\tau_1)\underset{JJ}{\Delta}{}^{kl}{}_{uj}(\gamma;\tau_1)\\ &+ 2\underset{JJ}{\overset{0}{\Omega}}{}^{vi}{}_{uv}(\gamma,\bar{\gamma};\tau_1)\underset{JJ}{\Delta}{}^{uv}{}_{uj}(\gamma;\tau_1),\end{aligned} \tag{4.264m}$$

$$\underset{JJ}{\overset{1/2}{\Omega}}{}^{vi}{}_{jk}(\gamma,\bar{\gamma};\tau_1) = \underset{JJ}{\overset{0}{\Omega}}{}^{vi}{}_{jk}(\gamma,\bar{\gamma};\tau_1), \tag{4.264n}$$

$$\underset{JJ}{\overset{1/2}{\Omega}}{}^{ij}{}_{uk}(\gamma,\bar{\gamma};\tau_1) = \underset{JP}{\Xi}{}^{ij}{}_l(\gamma,\bar{\gamma};\tau_1,\tau_0)\underset{PJ}{\Delta}{}^l{}_{uk}(\gamma;\tau_1) + 2\underset{JJ}{\Xi}{}^{ij}{}_{ul}(\gamma,\bar{\gamma};\tau_1)\left[\delta^l{}_k + 2\underset{JJ}{\Delta}{}^{ul}{}_{uk}(\gamma;\tau_1)\right]. \tag{4.264o}$$

This list of equations contains 31 nonzero components, which is fewer than the 50 that are required by the existence of a five-dimensional space of Killing vector fields. However, it does not contain all of the terms that are implied by expanding the products in equation (4.254)—such a naïve calculation gives nonzero values of $\underset{PP}{\overset{1/2}{\Omega}}{}^i{}_u(\gamma,\bar{\gamma};\tau_1)$ and $\underset{JP}{\overset{1/2}{\Omega}}{}^{ui}{}_u(\gamma,\bar{\gamma};\tau_1)$, which *must* be zero by equation (4.253). A careful inspection of these components, however, shows that they are zero:

$$\begin{aligned}\underset{PP}{\overset{1/2}{\Omega}}{}^i{}_u(\gamma,\bar{\gamma};\tau_1) &= \underset{PP}{\hat{\Delta}}{}^i{}_u(\gamma,\bar{\gamma};\tau_1) - \underset{PP}{\hat{\Delta}}{}^i{}_u(\gamma,\gamma;\tau_1) + \left[\delta^i{}_j + \underset{PP}{\hat{\Delta}}{}^i{}_j(\gamma,\gamma;\tau_1)\right]\underset{PP}{\overset{0}{\Omega}}{}^i{}_u(\gamma,\bar{\gamma};\tau_1)\\ &\quad + 2\underset{PJ}{\hat{\Delta}}{}^i{}_{uj}(\gamma,\gamma;\tau_1)\underset{JP}{\overset{0}{\Omega}}{}^{uj}{}_u(\gamma,\bar{\gamma};\tau_1)\\ &= {}_{\dot{x}}[\underset{PP}{\hat{\Delta}}{}^i{}_u]_j(\gamma,\gamma;\tau_1)\dot{\xi}^j(\tau_0) + {}_{\xi}\underset{PP}{\Theta}{}^i{}_{uj}(\gamma,\bar{\gamma};\tau_1)\xi^j(\tau_0)\\ &\quad - \frac{1}{\chi}\left[\delta^i{}_j + \underset{PP}{\hat{\Delta}}{}^i{}_j(\gamma,\gamma;\tau_1)\right]\left[\dot{\xi}^j(\tau_1) - \dot{\xi}^j(\tau_0)\right]\\ &\quad - 2\underset{PJ}{\hat{\Delta}}{}^i{}_{uj}(\gamma,\gamma;\tau_1)\left[\xi^j(\tau_0) - \xi^j(\tau_1) + (\tau_1-\tau_0)\dot{\xi}^j(\tau_1)\right]\\ &= \partial_{u_1}A_j{}^i(u_1,u_0)\xi^j(\tau_1) - \frac{1}{\chi}A_j{}^i(u_1,u_0)\dot{\xi}^j(\tau_1) + \frac{1}{\chi}\dot{\xi}^i(\tau_0)\\ &= -\partial_{u_0}A^i{}_j(u_0,u_1)\xi^j(\tau_1) - \frac{1}{\chi}\partial_{u_0}\left[(u_0-u_1)B^i{}_j(u_0,u_1)\right]\dot{\xi}^j(\tau_1) + \frac{1}{\chi}\dot{\xi}^i(\tau_0) = 0,\end{aligned} \tag{4.265a}$$

$$\begin{aligned}
\overset{1/2}{\underset{JP}{\Omega}}{}^{ui}{}_{u}(\gamma,\bar{\gamma};\tau_1) &= \underset{JP}{\hat{\Delta}}{}^{ui}{}_{u}(\gamma,\bar{\gamma};\tau_1) - \underset{JP}{\hat{\Delta}}{}^{ui}{}_{u}(\gamma,\gamma;\tau_1) + \left[\delta^i{}_j + 2\underset{JJ}{\hat{\Delta}}{}^{ui}{}_{uj}(\gamma,\gamma;\tau_1)\right] \overset{0}{\underset{PP}{\Omega}}{}^{uj}{}_{u}(\gamma,\bar{\gamma};\tau_1) \\
&\quad + \underset{JP}{\hat{\Delta}}{}^{ui}{}_{j}(\gamma,\gamma;\tau_1) \overset{0}{\underset{PP}{\Omega}}{}^{j}{}_{u}(\gamma,\bar{\gamma};\tau_1) \\
&= {}_{\dot{x}}[\underset{JP}{\hat{\Delta}}{}^{ui}{}_{u}]_j(\gamma,\gamma;\tau_1)\dot{\xi}^j(\tau_0) + {}_{\xi}\underset{JP}{\Theta}{}^{ui}{}_{uj}(\gamma,\bar{\gamma};\tau_1)\xi^j(\tau_0) \\
&\quad - \frac{1}{\chi}\underset{JP}{\hat{\Delta}}{}^{ui}{}_{j}(\gamma,\gamma;\tau_1)\left[\dot{\xi}^j(\tau_1) - \dot{\xi}^j(\tau_0)\right] \\
&\quad - \left[\delta^i{}_j + 2\underset{JJ}{\hat{\Delta}}{}^{ui}{}_{uj}(\gamma,\gamma;\tau_1)\right]\left[\xi^j(\tau_0) - \xi^j(\tau_1) + (\tau_1-\tau_0)\dot{\xi}^j(\tau_1)\right] \\
&= \partial_{u_1}\left[(u_1-u_0)B_j{}^i(u_1,u_0)\right]\xi^j(\tau_1) - \frac{1}{\chi}(u_1-u_0)B_j{}^i(u_1,u_0)\dot{\xi}^j(\tau_1) - \xi^i(\tau_0) \\
&= A^i{}_j(u_0,u_1)\xi^j(\tau_1) + (\tau_0-\tau_1)B^i{}_j(u_0,u_1)\dot{\xi}^j(\tau_1) - \xi^i(\tau_0) = 0. \qquad (4.265b)
\end{aligned}$$

The first equalities in these equations are simply a consequence of writing out equation (4.254) in terms of components. In the second equalities of these equations, we have used equations (4.245) and (4.246) to write the components of the affine transport holonomy in terms of $\xi^i(\tau_0)$, $\xi^i(\tau_1)$, and their derivatives, and we have also written out explicit expressions for the relevant components of $\Delta^A{}_B(\gamma;\tau_1)$, $\hat{\Delta}^A{}_B(\gamma,\gamma;\tau_1)$, and $\hat{\Delta}^A{}_B(\gamma,\bar{\gamma};\tau_1)$ using equations (4.259) and (4.261), which we then write in terms of the transverse Jacobi propagators in the third equalities using equations (4.257), (4.258), and (4.262). In the fourth equalities, we use the identities satisfied by the transverse Jacobi propagators in equations (4.216) and (4.217), and finally in the fifth equalities we use equation (4.240) and its derivative, but with τ_0 and τ_1 switched. Since $\overset{1/2}{\underset{PP}{\Omega}}{}^{i}{}_{u}(\gamma,\bar{\gamma};\tau_1)$ and $\overset{1/2}{\underset{JP}{\Omega}}{}^{ui}{}_{u}(\gamma,\bar{\gamma};\tau_1)$ vanish, equation (4.253) holds. Of the 50 remaining components, only the 31 given in equation (4.264) are nonzero. Note, however, that these 31 components are only determined by 12 functions, the independent components of the transverse Jacobi propagators.

4.3.3.3 Observables from a spinning test particle

We now consider the observables that can be determined from a spinning test particle. As remarked above, we will compute these observables perturbatively, using the expansion in equation (4.46), as we are not aware of ways to solve the fully nonlinear Mathisson-Papapetrou equations in plane wave spacetimes. To compute our results perturbatively, we use equation (4.177), adapting to a plane wave spacetime by using equation (4.225) for the parallel and Jacobi propagators. Assuming that $\xi^a \ell_a = 0$, we find that

$$\Upsilon^i{}_j(\tau_1, \tau_0) = -\chi\, {}_{\dot{x}}[\Upsilon^i{}_u]_j(\tau_1) = \int_{u_0}^{u_1} (u_1 - u_2) \underline{H}^i{}_k(u_1, u_2) \left(\underline{\mathcal{A}}^*\right){}^k{}_j(u_2), \tag{4.266a}$$

$${}_x[\Upsilon^{v'}{}_i]_j(\tau_0) = -\chi\, {}_{x\dot{x}}[\Upsilon^{v'}{}_u]_{ij} = \partial_{u_1} \underline{K}_{kj}(u_1, u_0) \Upsilon^k{}_i(\tau_1, \tau_0), \tag{4.266b}$$

$${}_{\dot{x}}[\Upsilon^{v'}{}_i]_j(\tau_0) = \frac{1}{\chi} \partial_{u_1} \left[(u_1 - u_0) \underline{H}_{kj}(u_1, u_0)\right] \Upsilon^k{}_i(\tau_1, \tau_0), \tag{4.266c}$$

$${}_{\dot{x}^2}[\Upsilon^{v'}{}_u]_{ij} = -\frac{1}{\chi}\, {}_{\dot{x}}[\Upsilon^{v'}{}_{(j}]_{i)}(\tau_0), \tag{4.266d}$$

$$\Psi^{v'}{}_{ij}(\tau_0) = -\chi\, {}_{\dot{x}}[\Psi^{v'}{}_{ui}]_j(\tau_0) = \int_{u_0}^{u_1} \mathrm{d}u_2 (u_1 - u_2) \left(\underline{\mathcal{A}}^*\right){}_{ki}(u_2) \partial_{u_2} \underline{K}^k{}_j(u_2, u_0). \tag{4.266e}$$

where $(\underline{\mathcal{A}}^*)_{ij}(u) \equiv \underline{\mathcal{A}}_{ik}(u) \underline{\epsilon}^k{}_j$ is defined as in appendix 4.A to this chapter.

At this point, let us focus on the observable $\Psi^{v'}{}_{ij}(\tau_0)$, which is an observable which does not seem able to be expressed solely in terms of sums and products of transverse Jacobi propagators and their derivatives. Using equations (4.216) and (4.266e), we can show that

$$\Psi^{v'}{}_{ij}(\tau_0) = -\partial_{u_0} \int_{u_0}^{u_1} \mathrm{d}u_2 (u_1 - u_2) \underline{K}_j{}^k(u_0, u_2) \left(\underline{\mathcal{A}}^*\right){}_{ki}(u_2). \tag{4.267}$$

The integrand does not appear to be in the form of a total derivative [unless $\underline{K}_j{}^k(u_0, u_1)$ and $\underline{K}_j{}^k(u_1, u_0)$ are proportional by a constant, which is not necessarily true]. As in section 4.3.3.1, we conclude by computing the proper time delay observable (but now for the spinning test particle):

$$\dot{\gamma}_{a'} \Delta\xi_S^{a'} = -\chi \left[L^{v'}{}_{ij}(\tau_0) \xi^i(\tau_0) \xi^j(\tau_0) + \Psi^{v'}{}_{ij}(\tau_0) \xi^i(\tau_0) s^i(\tau_0) + O(\boldsymbol{\xi}, \boldsymbol{s}^2)\right]. \tag{4.268}$$

Thus, $\Psi^{v'}{}_{ij}(\tau_0)$ also measures a sort of proper time delay observable [like $L^{v'}{}_{ij}(\tau_0)$ in section 4.3.3.1], except that it gives the dependence of this delay on spin in addition to separation.

4.3.3.4 Observables at second order in curvature

As in section 4.3.2.3, we now compute some parts of our persistent observables at second order in curvature. We do this both for general plane wave spacetimes and for the specific plane wave spacetime which we introduced in section 4.3.2.4. We focus on the quantities $L^{v'}{}_{ij}(\tau_0)$ in equation (4.243g) and $\Psi^{v'}{}_{ij}(\tau_0)$ in equation (4.266e) in this section. These results illustrate features of observables which can be computed from the transverse Jacobi propagators and their derivatives; other such observables are qualitatively similar.

The first observable which we compute is $L^{v'}{}_{ij}(\tau_0)$, which is a piece of the curve deviation observable defined by equation (4.8), with the value of this observable in arbitrary plane wave spacetimes given by equation (4.243g). Expanding this expression order-by-order, we find that it vanishes at zeroth order, whereas at first order we find it is

$$^{(1)}L^{v'}{}_{ij}(\tau_0) = \frac{1}{2}\partial_{u_1}\,{}^{(1)}\underline{K}_{(ij)}(u_1, u_0), \tag{4.269}$$

and at second order it is

$$^{(2)}L^{v'}{}_{ij}(\tau_0) = \frac{1}{2}\Bigg\{{}^{(1)}\underline{K}_{k(i}(u_1,u_0)\partial_{u_1}\,{}^{(1)}\underline{K}^k{}_{j)}(u_1,u_0) + \frac{1}{2}\partial_{u_1}\left[{}^{(1)}\underline{K}_{ik}(u_1,u_0)\,{}^{(1)}\underline{K}^k{}_j(u_1,u_0)\right] - \int_{u_0}^{u_1} \mathrm{d}u_2 \int_{u_0}^{u_2} \mathrm{d}u_3 \partial_{u_3}\,{}^{(1)}\underline{K}_{ik}(u_3,u_0)\,{}^{(1)}\underline{K}^k{}_j(u_3,u_0)\Bigg\}. \tag{4.270}$$

At second order, this observable is pure trace because it is symmetric and constructed from products of ${}^{(1)}\underline{K}^i{}_j(u', u)$, which is itself a symmetric and trace-free 2×2 matrix (assuming a vacuum plane wave spacetime). Using the wave profile (4.231), we have that

$$L^{v'}{}_{ij}(0)\Big|_{\tau_1 = \frac{2\pi n}{\omega\chi}} = -\frac{\pi\omega n\epsilon^2}{2}\left\{[\cos(2\phi) - 1]\,a^2 + 3\right\}\delta_{ij} + O(\epsilon^3). \tag{4.271}$$

Note that, like $\partial_{u_1} A^i{}_j(u_1, u_0)$, this observable vanishes at first order in ϵ.

The next observable which we consider is $\Psi^{v'}{}_{ij}(\tau_0)$, which is an observable from a spinning test particle which is defined by equation (4.46a). This observable is vanishing at first order by equation (4.266e), and this equation also implies that (at second order)

$$^{(2)}\Psi^{v'}{}_{ij}(\tau_0) = \int_{u_0}^{u_1} \mathrm{d}u_2 \Bigg\{\partial_{u_2}\,{}^{(1)}\underline{K}_{kl}(u_2,u_0)\partial_{u_2}\,{}^{(1)}\underline{K}^k{}_j(u_2,u_0) + \frac{1}{2}(u_1 - u_2)[\underline{A}(u_2), \partial_{u_2}\underline{K}(u_2,u_0)]_{lj}\Bigg\}\underline{\epsilon}^l{}_i. \tag{4.272}$$

As with the transverse Jacobi propagators, at second order there are both pieces that are pure trace and pieces that are antisymmetric (assuming a vacuum plane wave). However, because of the factor of $\underline{\epsilon}_{ab}$, it is the pure trace piece which only occurs when the wave is not linearly polarized, instead of the antisymmetric piece. Finally, we consider the wave profile in equation (4.231); we find that

$$\Psi^{v'}{}_{ij}(0)\Big|_{\tau_1 = \frac{2\pi n}{\omega\chi}} = -2\pi^2\omega n^2\epsilon^2 a\sqrt{1-a^2}\sin\phi\,\delta_{ij} + \frac{\pi\omega n\epsilon^2}{2}\left\{[\cos(2\phi) - 1]\,a^2 + 3\right\}\underline{\epsilon}_{ij} + O(\epsilon^3). \tag{4.273}$$

This expression has the same qualitative features as in the case of a general wave profile.

4.4 Discussion

In this chapter, we have introduced quantities that we called persistent gravitational wave observables, which are effects that share with the gravitational wave memory effect the feature of persistence after a burst of gravitational waves, but which are not necessarily associated with symmetries and conserved quantities at boundaries of spacetime. After reviewing many of the currently known persistent observables from the literature, we presented three new observables:

1. the difference between the separation of two accelerating curves from the result expected in flat space, which we called "curve deviation,"

2. the path dependence (or "holonomy") for two different methods for relating linear and angular momentum at different points (one inspired by how linear and angular momentum transform under a change of origin in flat space, and the other by the relationship between linear and angular momentum and Killing vectors), and

3. the difference between the initial and final separation, four-momentum, and spin of a spinning test particle that is initially comoving with some observer.

These observables measure the effects of the gravitational waves in a context where the spacetime transitions from a flat region, to a burst of gravitational waves, and then to another flat region.

We then provided the machinery with which one can calculate these observables in an arbitrary spacetime (which included reviewing the very powerful technique of covariant bitensors for understanding how tensor fields evolve along curves). Extending the results of [173, 174], we used these techniques to compute the holonomy with respect to an arbitrary connection around a variety of curves, as well as the evolution of the separation vector between two arbitrary worldlines. We then used these holonomies and the separation vector to compute our final results, which are in equations (4.143) for curve deviation, equations (4.156) and (4.168) for two different methods of relating angular momentum at different points, and equation (4.177) for the observables from a spinning test particle. Here, in order to make calculations tractable analytically, we made the simplifying assumption that the worldlines were closely-separated.

We also presented explicit expressions assuming that the curvature is small where these observables are being measured, so we may linearize in the spacetime curvature. This provides a connection to previous memory observables, which are typically discussed in this regime. These last results, where we linearize in the Riemann tensor, are important for discussing one possibility for measuring these persistent observables. Our results were given only in terms of various integrals, or alternatively, moments, of the Riemann tensor (and its derivatives) with respect to proper time. Moreover, in the limit where the gravitational waves are plane waves, these linearized results simplify even further, and they can be written entirely in terms of one, two, and three time integrals of the Riemann tensor, when there is no acceleration, and more time integrals, otherwise. As gravitational wave detectors effectively measure the Riemann tensor along their worldlines, these integrals of the Riemann tensor are (in principle) measurable. This would allow for our persistent observables to be measured indirectly.

A strength of our general results (not assuming weak curvature) are that they are not specialized to a particular spacetime. Our results are written in terms of the "fundamental bitensors," which are solutions to the equations of parallel transport (the parallel propagators) and linear geodesic deviation (the Jacobi propagators), which are known in a handful of spacetimes. In spacetimes where the geodesic equation has explicit solutions, these persistent observables can even be computed without assuming that the neighboring worldlines are closely-separated.

As an example of such a spacetime, we considered these observables in nonlinear, exact plane wave spacetimes. These spacetimes possess an important set of two functions (and their derivatives), which we refer to as transverse Jacobi propagators. Many of the geometric properties of these spacetimes, such as Killing vectors and solutions to the geodesic equation, can be written in terms of these functions. Our primary result is that many parts of the persistent observables in this chapter can be determined just from the values of these functions and their derivatives. We found in our linear, plane wave results that many parts of our observables could be written in terms of a small number of functions, but the fact that this statement also holds in the nonlinear context is unexpected.

The main utility of this result is that only the transverse Jacobi propagators are necessary to determine the values of many of our persistent observables. That is, although the persistent observables we have defined encompass a large number of interesting physical effects, many of these effects are determined by just a small number of functions. These functions, in turn, can be determined by the displacement memory observable (which gives the transverse Jacobi propagators directly) and the relative velocity observable (which gives their derivatives).

Example spacetimes in which to consider the persistent observables in this chapter are not limited to plane waves. In particular, one could consider the more general class of "pp-wave" spacetimes, which are a generalization of plane wave spacetimes where the planar wavefronts are not homogeneous, as $\underline{\mathcal{A}}_{ij}$ is also a function of x^i. Such spacetimes have Jacobi and parallel propagators that one can calculate using a procedure that is similar to the one we carried out in this chapter, but the geodesic equation does not have exact solutions, nor are the transverse Jacobi propagators solely functions of u. In plane wave spacetimes, one only needed to determine the transverse Jacobi propagators along a given timelike geodesic in order to compute persistent observables, but in pp-wave spacetimes one would need to determine them along all timelike geodesics.

Finally, a natural regime to study persistent gravitational wave observables is near future null infinity; of particular interest are their falloffs in $1/r$ near null infinity. Here, the contexts that are relevant for studying persistent observables are spacetimes that possess two nonradiative regions that are separated by a radiative region. As the two nonradiative regions are no longer flat, it is possible that the observables in this chapter will also measure parts of the spacetime curvature not related to the gravitational waves, and so will not qualify as persistent gravitational wave observables in this context. In a future future, we plan to discuss the persistent gravitational wave observables that arise near null infinity.

Appendix

4.A Dualization of Arbitrary Tensors

Following Penrose and Rindler [129, 130] we define (in four dimensions) left and right duals of tensors acting on either the first or last two indices:

$$({}^*Z)_{abc_1\cdots c_s} \equiv \frac{1}{2}\epsilon_{abde} Z^{de}{}_{c_1\cdots c_s}, \qquad (Z^*)_{a_1\cdots a_s bc} \equiv \frac{1}{2} Z_{a_1\cdots a_s}{}^{de}\epsilon_{debc}. \tag{4.274}$$

In addition to this standard definition, they define another type of dual, which acts on the first or last indices,

$$({}^{\dagger}Z)_{abcd_1\cdots d_s} \equiv \epsilon_{eabc} Z^e{}_{d_1\cdots d_s}, \qquad (Z^{\dagger})_{a_1\cdots a_s bcd} \equiv Z_{a_1\cdots a_s}{}^e \epsilon_{ebcd}, \tag{4.275}$$

and a dual acting on the first or last three indices,

$$({}^{\ddagger}Z)_{ab_1\cdots b_s} \equiv \frac{1}{6}\epsilon_{cdea} Z^{cde}{}_{b_1\cdots b_s}, \qquad (Z^{\ddagger})_{a_1\cdots a_s b} \equiv \frac{1}{6} Z_{a_1\cdots a_s}{}^{cde}\epsilon_{cdeb}. \tag{4.276}$$

With these definitions, we have that

$$({}^{**}Z)_{abc_1\cdots c_s} = -Z_{[ab]c_1\cdots c_s}, \qquad (Z^{**})_{a_1\cdots a_s bc} = -Z_{a_1\cdots a_s[bc]}, \tag{4.277}$$

$$({}^{\ddagger\dagger}Z)_{ab_1\cdots b_s} = Z_{ab_1\cdots b_s}, \qquad (Z^{\dagger\ddagger})_{a_1\cdots a_s b} = Z_{a_1\cdots a_s b}, \tag{4.278}$$

$$({}^{\dagger\ddagger}Z)_{abcd_1\cdots d_s} = Z_{[abc]d_1\cdots d_s}, \qquad (Z^{\ddagger\dagger})_{a_1\cdots a_s bcd} = Z_{a_1\cdots a_s[bcd]}. \tag{4.279}$$

In four dimensions, these are the only useful definitions of duals of arbitrary tensors. In two dimensions, which is relevant to the case of plane wave spacetimes, it is also sensible to define

$$({}^{*}A)_{ab_1\cdots b_s} \equiv \epsilon_{ca} A^c{}_{b_1\cdots b_s}, \qquad (A^{*})_{a_1\cdots a_s b} \equiv A_{a_1\cdots a_s}{}^c \epsilon_{cb}, \tag{4.280}$$

and one has that

$$({}^{**}A)_{ab_1\cdots b_s} = -A_{ab_1\cdots b_s}, \qquad (A^{**})_{a_1\cdots a_s b} = -A_{a_1\cdots a_s b}, \tag{4.281}$$

assuming a Riemannian signature for the two-dimensional metric (as is relevant to the discussion in this chapter). A particularly useful special case is when one is considering 2×2 matrices. One easily show that

$$({}^{*}A^{*})^a{}_b = A_c{}^c \delta^a{}_b - A_b{}^a, \tag{4.282}$$

and so

$$(A^{*})^a{}_b = \begin{cases} ({}^{*}A)^a{}_b & A_{ab} \text{ symmetric, trace-free} \\ -({}^{*}A)^a{}_b & A_{ab} \text{ pure trace or antisymmetric} \end{cases}. \tag{4.283}$$

As such, we find that if A_{ab} and B_{ab} are both symmetric, trace-free, then

$$A^a{}_c (B^{*})^c{}_b = A^a{}_c \epsilon_d{}^c B_{db} = -({}^{*}A)^a{}_c B^c{}_b, \tag{4.284}$$

which implies that $A_{ac}B^c{}_b$ is a sum of a pure trace and an antisymmetric term—in particular, if $A_{ab} = B_{ab}$, then it must be pure trace.

4.B Algebraic Decomposition of Holonomies

In this section, we present a method of reducing the holonomy observable in section 4.2.2.2 into more manageable pieces. Our method is purely algebraic and applies to general matrices on the linear and angular momentum bundle. Consider first any matrix $A^A{}_B$, which we break into components as in equation (4.23). We now perform an algebraic decomposition of each of these pieces:

$$\underset{PP}{A}{}^a{}_b \equiv \underset{[PP]}{A}{}^a{}_b + \underset{\langle PP\rangle}{A}{}^a{}_b + \frac{1}{4}\underset{PP}{A}\delta^a{}_b, \tag{4.285a}$$

$$\underset{PJ}{A}{}^a{}_{bc} \equiv 2\underset{PJ}{A}{}_{[b}\delta^a{}_{c]} + \big({}^\dagger\underset{\ddagger PJ}{A}\big)^a{}_{bc} + \underset{\langle PJ\rangle}{A}{}^a{}_{bc}, \tag{4.285b}$$

$$\underset{JP}{A}{}^{ab}{}_c \equiv 2\underset{JP}{A}{}^{[a}\delta^{b]}{}_c + \big({}^\dagger\underset{\ddagger JP}{A}\big)^{ab}{}_c + \underset{\langle JP\rangle}{A}{}^{ab}{}_c, \tag{4.285c}$$

$$\underset{JJ}{A}{}^{ab}{}_{cd} \equiv 2\delta^{[a}{}_{[c}\underset{JJ}{A}{}^{b]}{}_{d]} + \underset{[JJ]}{A}{}^{ab}{}_{cd} + \underset{\langle JJ\rangle}{A}{}^{ab}{}_{cd} + \underset{*JJ}{A}\epsilon^{ab}{}_{cd}. \tag{4.285d}$$

We also decompose $\underset{JJ}{A}{}^a{}_b$ in the second-to-last line as

$$\underset{JJ}{A}{}^a{}_b \equiv \underset{[JJ]}{A}{}^a{}_b + \underset{\langle JJ\rangle}{A}{}^a{}_b + \frac{1}{4}\underset{JJ}{A}\delta^a{}_b. \tag{4.285e}$$

These algebraically irreducible pieces have the following properties:

1. $\underset{[PP]}{A}{}^a{}_b$ and $\underset{[JJ]}{A}{}^a{}_b$ are antisymmetric, and have 6 independent components each;

2. $\underset{\langle PP\rangle}{A}{}^a{}_b$ and $\underset{\langle JJ\rangle}{A}{}^a{}_b$ are symmetric and trace-free, and have 9 independent components each;

3. $\underset{\langle JP\rangle}{A}{}^{ab}{}_c$ and $\underset{\langle PJ\rangle}{A}{}^a{}_{bc}$ are trace-free on all indices and satisfy

$$\underset{\langle PJ\rangle}{A}{}_{[abc]} = \underset{\langle JP\rangle}{A}{}_{[abc]} = 0, \tag{4.286}$$

   implying they have 16 independent components each;

4. $\underset{[JJ]}{A}{}^{ab}{}_{cd}$ is trace-free on all indices and antisymmetric on interchange of the first two and last two indices, so it has 9 independent components; and

5. $\underset{\langle JJ\rangle}{A}{}^{ab}{}_{cd}$ is trace-free on all indices, symmetric on interchange of the first two and last two indices, and satisfies

$$\underset{\langle JJ\rangle}{A}{}_{[abcd]} = 0, \tag{4.287}$$

   giving it 10 independent components.

The following results show how to construct the algebraically irreducible pieces from the full matrix $A^A{}_B$:

$$\underset{[xx]}{A}{}_{ab} = \underset{xx}{A}{}_{[ab]}, \qquad \underset{xx}{A} = \underset{xx}{A}{}^a{}_a, \qquad \underset{\langle xx\rangle}{A}{}_{ab} = \underset{xx}{A}{}_{(ab)} - \frac{1}{4} g_{ab}\underset{xx}{A}, \tag{4.288a}$$

$$\underset{JJ}{A}{}^a{}_b = \underset{JJ}{A}{}^{ac}{}_{bc} + \frac{1}{6}\underset{JJ}{A}{}^{cd}{}_{cd}\delta^a{}_b, \tag{4.288b}$$

$$\underset{PJ}{A}{}_a = -\frac{1}{3}\underset{PJ}{A}{}^b{}_{ba}, \qquad \underset{JP}{A}{}^a = -\frac{1}{3}\underset{JP}{A}{}^{ab}{}_b, \qquad \underset{\ddagger xy}{A}{}_a = \left({}^{\ddagger}\underset{xy}{A}\right)_a, \tag{4.288c}$$

$$\underset{\langle PJ\rangle}{A}{}^a{}_{bc} = \underset{PJ}{A}{}^a{}_{bc} - \underset{PJ}{A}{}_{[b}\delta^a{}_{c]} + \epsilon^a{}_{bcd}\underset{\ddagger PJ}{A}{}^d, \qquad \underset{\langle JP\rangle}{A}{}^{ab}{}_c = \underset{JP}{A}{}^{ab}{}_c - \underset{JP}{A}{}^{[a}\delta^{b]}{}_c + \epsilon^{ab}{}_{cd}\underset{\ddagger JP}{A}{}^d, \tag{4.288d}$$

$$\underset{[JJ]}{A}{}^{ab}{}_{cd} = \frac{1}{2}\left(\underset{JJ}{A}{}^{ab}{}_{cd} - \underset{JJ}{A}{}_{cd}{}^{ab}\right) - 2\delta^{[a}{}_{[c}\underset{[JJ]}{A}{}^{b]}{}_{d]}, \tag{4.288e}$$

$$\underset{*JJ}{A} = -\frac{1}{24}\epsilon^{abcd}\underset{JJ}{A}{}_{abcd}, \tag{4.288f}$$

$$\underset{\langle JJ\rangle}{A}{}^{ab}{}_{cd} = \frac{1}{2}\left(\underset{JJ}{A}{}^{ab}{}_{cd} + \underset{JJ}{A}{}_{cd}{}^{ab}\right) - 2\delta^{[a}{}_{[c}\underset{\langle JJ\rangle}{A}{}^{b]}{}_{d]} - \frac{1}{2}\delta^{[a}{}_{[c}\delta^{b]}{}_{d]}\underset{JJ}{A} - \underset{*JJ}{A}\epsilon^{ab}{}_{cd}, \tag{4.288g}$$

where $x = P, J$ and $y \neq x$.

There are two main uses of this decomposition. The first is that many of these pieces have a physically relevant meaning. For example, assuming that $J^{ab} = 0$, then $\underset{[PP]}{A}{}^a{}_b$, $\underset{\langle PP\rangle}{A}{}^a{}_b$, and $\underset{\langle PP\rangle}{A}$ can be understood as an infinitesimal rotation, shear, and expansion of P^a, respectively (the latter two transformations change the rest mass $P^a P_a$). As another example, $\underset{JP}{A}{}^a$ is the term that contributes to the change in J^{ab} in flat spacetime from a change of origin.

The second main use of this decomposition is that certain of these irreducible pieces may vanish for particular matrices; this could make it easier to compute the number of independent components that these matrices have. For example, in the case where $A^A{}_B = \overset{0}{\Omega}{}^A{}_B(\gamma, \bar{\gamma}; \tau')$, we can easily see from equation (4.158) that the only nonzero pieces are $\underset{[PP]}{A}{}^a{}_b = \underset{[JJ]}{A}{}^a{}_b$ and $\underset{JP}{A}{}^a$; thus, the holonomy has only 10 independent components. Similarly, if we set $A^A{}_B = \overset{\varkappa}{R}{}^A{}_{Bcd}\dot{\gamma}^c\dot{\bar{\gamma}}^d$ (an infinitesimal version of the holonomy for arbitrary $\varkappa$), we can easily show from equation (4.151) and the symmetries of $\overset{\varkappa}{K}_{abcd}$ in equation (4.149) that

$$\underset{JP}{A}{}^{ab}{}_c = 0, \qquad \underset{PP}{A} = \underset{JJ}{A} = \underset{*JJ}{A} = 0, \qquad \underset{\ddagger PJ}{A}{}^a = 0. \tag{4.289}$$

This matrix then can have at most 69 independent components (it has fewer, but the algebraic decomposition only gives us an upper bound). For the general case of the holonomy for arbitrary

$\varkappa$ around a narrow loop, the algebraic decomposition gives no additional information about the number of independent components.

Chapter 5

Angular Momentum in Einstein-Maxwell Theory

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We conclude this dissertation with a discussion of angular momentum in the theory of general relativity coupled to electromagnetism, which we simply refer to as *Einstein-Maxwell theory*.

This discussion is partially motivated by a strange feature of electromagnetism in flat spacetime. One typically thinks of fluxes of conserved currents that are associated with conserved quantities as depending only on the radiative degrees of freedom—that is, those that fall off as $1/r$ and that vanish if there is no radiation present. However, fluxes do not always depend purely on radiative degrees of freedom. In particular, the flux of angular momentum (when calculated using the stress-energy tensor) depends as well on *Coulombic* ($1/r^2$) degrees of freedom, which *do not* vanish when there is no radiation present [19, 20]. The terms that contain these Coulombic degrees of freedom are relevant for some physical systems: in fact, *all* of the angular momentum radiated by a charged spinning sphere with variable angular velocity is due to these terms [39].

This dependence on Coulombic degrees of freedom arises for the flux of a conserved current (which we call "angular momentum") that is constructed from the stress-energy tensor. However, as remarked in the introduction, there are other conserved currents for electromagnetic fields that are also naturally associated with Killing symmetries. In this chapter, we consider two currents, the *Noether current* and the *canonical current*, which are both constructed from the Maxwell Lagrangian. The former is the natural conserved current associated with Killing symmetries through

Noether's theorem, and the latter is related to the symplectic products discussed above in chapter 3. As with the current defined by the stress-energy tensor, each of these currents is conserved, defining fluxes of energy and linear momentum (for time or space translation Killing fields), as well as angular momentum (associated with rotational Killing fields). Moreover, for all Killing vector fields in flat spacetime, these fluxes only depend on radiative degrees of freedom.

The Noether and canonical currents associated with angular momentum differ from the stress-energy current by total derivatives, and it is entirely in these total derivatives that the dependence on Coulombic fields arises. When evaluating fluxes through regions of future null infinity, these total derivatives yield "boundary terms" evaluated along cross-sections of null infinity that depend on the Coulombic degrees of freedom. These boundary terms do not vanish when one takes the limits of these cross-sections to timelike and spatial infinity—in particular, they are nonzero when there is electromagnetic memory (see [31]). Thus, *a priori*, it is not obvious which (if any) of these currents defines the "correct" notion of angular momentum flux through null infinity for electromagnetic fields on a flat background.

In this chapter, we first show that the above considerations generalize to the asymptotic symmetries in electromagnetism on any non-dynamical, asymptotically flat background spacetime. In particular, one can define the fluxes through null infinity using any of the aforementioned currents associated with the generators of the Bondi-Metzner-Sachs (BMS) algebra. We find that the Noether and canonical currents define fluxes associated with *all* BMS symmetries, and these fluxes are completely determined by the radiative degrees of freedom of the electromagnetic fields. However, the flux of the stress-energy current that is associated with asymptotic Lorentz symmetries depends also on the Coulombic part via a "boundary term" exactly as in Minkowski spacetime. Furthermore, none of these fluxes can be written as the change of a charge computed purely on cross-sections of null infinity. Thus, working purely on null infinity, none of these fluxes can be interpreted as the change in "energy" or "angular momentum" on cross-sections of null infinity.

To investigate this issue in more detail, we then consider Einstein-Maxwell theory, in which the metric is now a dynamical field. Unlike electromagnetism on a non-dynamical background, Einstein-Maxwell theory is diffeomorphism covariant. Thus, we can apply the general prescription of Wald and Zoupas [178] to define charges $\mathcal{Q}$ (on any cross-section of null infinity) and their fluxes

$\mathcal{F}$ (which are the change in charges $\mathcal{Q}$ through any region of null infinity) associated with the BMS symmetries at null infinity.

We show that if one defines charges for the BMS symmetries using the same expression as in vacuum general relativity [$\mathcal{Q}_{\text{GR}}$ in equation (5.133)], then the contribution to the fluxes of these charges due to electromagnetic fields is given by the stress-energy current. Consequently, the fluxes associated with asymptotic Lorentz symmetries, such as angular momentum, are not purely radiative but depend also on the Coulombic parts of the electromagnetic fields. However, if we instead define charges by applying the Wald-Zoupas prescription to the full Einstein-Maxwell theory, we find that the charges are of the form $\mathcal{Q} = \mathcal{Q}_{\text{GR}} + \mathcal{Q}_{\text{EM}}$, where the additional contribution $\mathcal{Q}_{\text{EM}}$ [equation (5.137)] depends on the electromagnetic fields. We show that the flux $\mathcal{F}$ of the full Wald-Zoupas charge across any region of null infinity is completely determined by the radiative degrees of freedom of both the gravitational and electromagnetic fields at null infinity. The contribution of the electromagnetic fields to this Wald-Zoupas flux is, in fact, given by the Noether current and *not* the stress-energy current. In addition, the Wald-Zoupas flux $\mathcal{F}$ through *all* of null infinity defines a Hamiltonian generator associated with the BMS symmetries on the radiative phase space of Einstein-Maxwell theory at null infinity.

We further show that the additional contribution $\mathcal{Q}_{\text{EM}}$ vanishes for supertranslations and does not contribute to the supermomentum charges associated with supertranslation symmetries. In particular, the supermomentum charge is given by the usual formula $\mathcal{Q}_{\text{GR}}$ as in vacuum GR, and the supermomentum flux gets an additional (purely radiative) contribution from the electromagnetic fields which is equal to the flux determined by the stress-energy or Noether current (as they are equal for supertranslations). If one considers the Kerr-Newman solution, the additional contribution $\mathcal{Q}_{\text{EM}}$ vanishes for Lorentz symmetries as well. However, for non-stationary solutions of Einstein-Maxwell theory, $\mathcal{Q}_{\text{EM}}$ is generically non-vanishing for Lorentz symmetries. Thus, in general, the contribution due to electromagnetic fields to the Wald-Zoupas flux of Lorentz charges, for example angular momentum, is not given by the flux of stress-energy but instead by the Noether current flux.

The rest of the chapter is organized as follows. We first consider electromagnetism on a fixed background in section 5.1. In section 5.1.1, we review the natural currents of electromagnetism associated with vector fields in a non-dynamical spacetime which are conserved for Killing vector fields. In section 5.1.2.2, we derive useful properties of the asymptotic symmetries of electromagnetism on a fixed background. In section 5.1.3, we consider the limits of these currents to null infinity for BMS vector fields, which need not be exact Killing vector fields, and define the corresponding fluxes associated with the BMS symmetries.

We then consider dynamical backgrounds. In section 5.2, we consider Einstein-Maxwell theory, analyze its symplectic current, and review the asymptotic conditions at null infinity. Some properties of stationary solutions in Einstein-Maxwell theory at null infinity are presented in section 5.2.3. In section 5.3, we use the Wald-Zoupas prescription to define charges and fluxes associated with the BMS algebra in Einstein-Maxwell theory. We review the essential ingredients of the Wald-Zoupas prescription in section 5.3.1 and compute the charges and fluxes for Einstein-Maxwell theory at null infinity in section 5.3.2. We provide examples of computing these charges in section 5.3.2.2, both in the Kerr-Newman spacetime and for a charged spinning sphere in the Minkowski spacetime. We end with section 5.4 by discussing our main results and their implications.

We use the following conventions not specified in the introduction. Contraction of vectors into the first index of a differential form is denoted by "$\cdot$", for example $(X \cdot \theta)_{ab} \equiv X^c \theta_{cab}$ for a vector field X^a and a 3-form θ_{abc}. We also adopt the following convention related to the usual conformal construction of null infinity (see section 5.1.2): fields in the physical spacetime are denoted with hats, while the corresponding unphysical quantities are unhatted. For example, $\hat{g}_{ab}$ is the physical spacetime metric while g_{ab} is the metric in the unphysical (conformally-completed) spacetime. Moreover, all quantities with hats have their indices raised and lowered with the physical metric $\hat{g}_{ab}$, and all quantities without hats have their indices raised and lowered with the unphysical metric g_{ab}. Finally, for differential forms, we denote the physical volume element by $\hat{\boldsymbol{\epsilon}}_4$, or $\hat{\epsilon}_{abcd}$ when we use indices—we adopt this convention as there are several volume elements, of different dimension, that arise in this chapter. Quantities with hats will have their Hodge dual taken with the physical volume element $\hat{\boldsymbol{\epsilon}}_4$, and those without hats will have their Hodge dual taken with $\boldsymbol{\epsilon}_4$.

We also use the following terminology for the charges and fluxes associated with the symmetry algebra at null infinity. Quantities associated with asymptotic symmetries evaluated as integrals over cross-sections $S \cong \mathbb{S}^2$ of null infinity will be called "charges", while those evaluated as an integral over a portion $\Delta\mathscr{I}$ of null infinity bounded by two cross-sections will be called "fluxes". In general, fluxes need not be the difference of any charges on the two bounding cross-sections, but the Wald-Zoupas fluxes (defined in section 5.3.1) are the change of the Wald-Zoupas charges. When certain conditions are satisfied, the fluxes given by the Wald-Zoupas prescription can also be considered as Hamiltonian generators on the phase space at null infinity [see the discussion below equation (5.123)].

5.1 Angular Momentum in Electromagnetism on a Fixed Background

In this section, we study the properties of electromagnetism on a fixed background, and in particular conserved currents in this theory.

We begin with a review of the theory of electromagnetism. The dynamical field of electromagnetism is given by a vector potential, which is most naturally treated as a connection on a $U(1)$-principal bundle over spacetime. The analysis in this chapter can then be performed directly on this principal bundle [133]. Since this approach would need considerable additional formalism, we will instead treat the vector potential as a 1-form $\hat{A}_a$ on spacetime which is obtained from the connection by making an (arbitrary) choice of gauge. The Faraday tensor $\hat{F}_{ab}$ is then

$$\hat{F}_{ab} \equiv 2\hat{\nabla}_{[a}\hat{A}_{b]} = (\mathrm{d}\hat{A})_{ab}. \tag{5.1}$$

We now consider the transformations of the vector potential under both gauge transformations parametrized by a function $\hat{\lambda}$ and diffeomorphisms generated by a vector field $\hat{X}^a$, which we collectively denote by $\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{X}}, \hat{\lambda})$. Understanding these transformations is necessary for the construction of currents in this section. The infinitesimal change in the vector potential under these transformations is given by

$$\delta_{\hat{\xi}}\hat{A}_a = \pounds_{\hat{X}}\hat{A}_a + \hat{\nabla}_a\hat{\lambda} = \hat{X}^b\hat{F}_{ba} + \hat{\nabla}_a\left(\hat{X}^b\hat{A}_b + \hat{\lambda}\right). \tag{5.2}$$

One may now ask how equation (5.2) behaves under the choice of gauge for $\hat{A}_a$. For a *fixed* transformation parametrized by $\hat{\boldsymbol{\xi}}$, its representation in terms of a vector field $\hat{X}^a$ and a gauge transformation $\hat{\lambda}$ should depend on this gauge choice. Let $\hat{A}'_a = \hat{A}_a + \hat{\nabla}_a\hat{\Lambda}$ be another vector potential related to $\hat{A}_a$ by a gauge transformation $\hat{\Lambda}$. For a fixed $\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{X}}, \hat{\lambda})$, let the new representatives under the gauge transformation by $\hat{\Lambda}$ be $\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{X}}', \hat{\lambda}')$. Since $\hat{\boldsymbol{\xi}}$ is fixed, its action on the vector potentials must be independent of the choice of gauge; that is, $\delta_{\hat{\xi}}\hat{A}'_a = \delta_{\hat{\xi}}\hat{A}_a$. Evaluating this equation, we find

$$\pounds_{\hat{X}'}\hat{A}_a + \hat{\nabla}_a\hat{\lambda}' + \hat{\nabla}_a\pounds_{\hat{X}'}\hat{\Lambda} = \pounds_{\hat{X}}\hat{A}_a + \hat{\nabla}_a\hat{\lambda}. \tag{5.3}$$

This implies that, under a change of gauge by $\hat{\Lambda}$, the representation of a fixed transformation $\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{X}}, \hat{\lambda}) = (\hat{\boldsymbol{X}}', \hat{\lambda}')$ changes as

$$\hat{X}'^a = \hat{X}^a, \qquad \hat{\lambda}' = \hat{\lambda} - \pounds_{\hat{X}}\hat{\Lambda}. \tag{5.4}$$

Consequently, the notion of a pure gauge transformation $\hat{\boldsymbol{\xi}} = (0, \hat{\lambda})$ is well-defined independently of the choice of gauge $\hat{\Lambda}$, but a "pure diffeomorphism" $\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{X}}, 0)$ is not. This is analogous to the structure of the BMS algebra noted in section 5.1.2.2. Note also that

$$\hat{\lambda}' + \hat{X}'^a\hat{A}'_a = \hat{\lambda} + \hat{X}^a\hat{A}_a \tag{5.5}$$

is invariant under changes of gauge.[1] This can also be seen directly from the requirement that equation 5.2 be preserved under a gauge transformation, assuming that $\hat{X}'^a = \hat{X}^a$.

In the remainder of this section, we discuss in detail three currents (associated with vector fields) that occur in electromagnetism on a fixed, non-dynamical background spacetime: the canonical, stress-energy, and Noether currents. We show that, when the vector field is a Killing field of the background metric, each of these currents is conserved and they differ by "boundary" terms. We then review the asymptotic structure of the non-dynamical background and the asymptotic behavior of the electromagnetic field. Many results of this section will carry over to our discussion of Einstein-Maxwell theory later in this chapter. Using the asymptotic structure of this theory, we then carefully analyze the fluxes through $\mathscr{I}$ defined by each of these currents, assuming that these vector fields are asymptotic symmetries in the BMS algebra.

[1] On a principal bundle, $\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{X}}, \hat{\lambda})$ is a vector field on the bundle and equation (5.2) gives the Lie derivative of the connection with respect to $\hat{\boldsymbol{\xi}}$. The Lie algebra of such vector fields also has the structure of a semidirect sum of diffeomorphisms with the Lie ideal of gauge transformations [133]. The invariant in equation (5.5) is then the vertical part of $\hat{\boldsymbol{\xi}}$ on the bundle.

5.1.1 Conserved currents

The conserved currents for electromagnetism which we will consider in this chapter arise from the Lagrangian formulation of this theory. The Lagrangian 4-form of electromagnetism is given by

$$\boldsymbol{L}_{\rm EM} \equiv \hat{\boldsymbol{\epsilon}}_4 \left(-\frac{1}{16\pi}\hat{F}^2\right), \tag{5.6}$$

where $\hat{F}^2 \equiv \hat{g}^{ac}\hat{g}^{bd}\hat{F}_{ab}\hat{F}_{cd}$ and the metric is considered to be a non-dynamical field. One can also consider the electromagnetic field coupled to a charged source current of compact support. In flat spacetime, such source currents are necessary to have a non-vanishing Coulombic part of the electromagnetic field. Of course, there are asymptotically flat spacetimes which are solutions of the source-free Maxwell equations and have a non-vanishing Coulombic part, without the need to introduce external sources: for example, the Kerr-Newman spacetime describing a charged, spinning black hole. Since we are mostly concerned with the behavior at null infinity, a source current of compact support does not change our main analysis. As such, we will not explicitly consider such a source current.

Varying the Lagrangian with respect to the dynamical field $\hat{A}_a$ gives

$$\delta\boldsymbol{L}_{\rm EM} = \hat{\boldsymbol{\epsilon}}_4 \left[\frac{1}{4\pi}\left(\hat{\nabla}_b\hat{F}^{ba}\right)\delta\hat{A}_a - \frac{1}{4\pi}\hat{\nabla}_b\left(\hat{F}^{ba}\delta\hat{A}_a\right)\right], \tag{5.7}$$

which yields the Maxwell equations

$$\hat{\nabla}_b\hat{F}^{ba} = 0, \tag{5.8}$$

as well as a "boundary term" corresponding to the symplectic potential 3-form:

$$\boldsymbol{\theta}_{\rm EM}[\hat{\boldsymbol{A}};\delta\hat{\boldsymbol{A}}] \equiv {}^\star\hat{\boldsymbol{v}}_{\rm EM}[\hat{\boldsymbol{A}};\delta\hat{\boldsymbol{A}}], \tag{5.9}$$

where

$$\hat{v}^a_{\rm EM}[\hat{\boldsymbol{A}};\delta\hat{\boldsymbol{A}}] \equiv -\frac{1}{4\pi}\hat{F}^{ab}\delta\hat{A}_b. \tag{5.10}$$

The symplectic current 3-form is then defined as

$$\boldsymbol{\omega}_{\rm EM}[\hat{\boldsymbol{A}};\delta_1\hat{\boldsymbol{A}},\delta_2\hat{\boldsymbol{A}}] \equiv \delta_1\boldsymbol{\theta}_{\rm EM}[\hat{\boldsymbol{A}};\delta_2\hat{\boldsymbol{A}}] - \delta_2\boldsymbol{\theta}_{\rm EM}[\hat{\boldsymbol{A}};\delta_1\hat{\boldsymbol{A}}] = {}^\star\hat{\boldsymbol{w}}_{\rm EM}[\delta_1\hat{\boldsymbol{A}},\delta_2\hat{\boldsymbol{A}}], \tag{5.11}$$

where

$$\hat{w}^a_{\rm EM}[\delta_1\hat{\boldsymbol{A}},\delta_2\hat{\boldsymbol{A}}] \equiv -\frac{1}{4\pi}\left[\delta_1\hat{F}^{ab}\delta_2\hat{A}_b - (1\leftrightarrow 2)\right]. \tag{5.12}$$

From this symplectic current, we construct the *canonical current* for a transformation of the vector potential [equation (5.2)] generated by $\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{X}}, \hat{\lambda})$. *A priori*, one may naively expect the canonical current to involve two variations of the vector potential. However, since the Maxwell equations are linear, the situation simplifies: consider a one-parameter family of vector potentials $\hat{A}_a(\epsilon) \equiv (1+\epsilon)\hat{A}_a$. This entire family satisfies the Maxwell equations if $\hat{A}_a$ satisfies the Maxwell equations, and the variation of this family of solutions $\delta_\epsilon \hat{A}_a \equiv \frac{\mathrm{d}}{\mathrm{d}\epsilon}\hat{A}_a(\epsilon)|_{\epsilon=0}$ is equal to the vector potential $\hat{A}_a$. Therefore, for a given symmetry $\hat{\xi} \equiv (\hat{X}^a, \hat{\lambda})$, where $\hat{X}^a$ is any vector field and $\hat{\lambda}$ is a gauge transformation, we can define the *canonical current* by

$$\boldsymbol{J}_{\mathrm{C}}[\hat{\boldsymbol{\xi}}] \equiv \boldsymbol{\omega}_{\mathrm{EM}}[\hat{\boldsymbol{A}}; \delta_\epsilon \hat{\boldsymbol{A}}, \delta_{\hat{\xi}} \hat{\boldsymbol{A}}] \equiv {}^\star \hat{\boldsymbol{\jmath}}_{\mathrm{C}}[\hat{\boldsymbol{\xi}}], \tag{5.13}$$

where

$$\hat{\jmath}^a_{\mathrm{C}}[\hat{\boldsymbol{\xi}}] = -\frac{1}{4\pi}\left[\hat{F}^{ab}\left(\pounds_{\hat{X}}\hat{A}_b + \hat{\nabla}_b\hat{\lambda}\right) - \hat{g}^{ac}\hat{g}^{bd}\hat{A}_b \pounds_{\hat{X}}\hat{F}_{cd}\right]. \tag{5.14}$$

To define the stress-energy and Noether currents, we also need to vary the Maxwell Lagrangian with respect to the metric $\hat{g}_{ab}$.[2] In particular, by varying the Lagrangian with respect to the non-dynamical metric $\hat{g}_{ab}$ we find the Maxwell stress-energy tensor $\hat{T}^{ab}$:

$$\delta_{\hat{g}}\boldsymbol{L}_{\mathrm{EM}} = \frac{1}{2}\hat{\boldsymbol{\epsilon}}_4 \hat{T}^{ab}\delta\hat{g}_{ab}, \tag{5.15}$$

where

$$\hat{T}^{ab} \equiv \frac{1}{4\pi}\left(\hat{F}^{ac}\hat{F}^b{}_c - \frac{1}{4}\hat{g}^{ab}\hat{F}^2\right). \tag{5.16}$$

The associated current, the *stress-energy current* for some vector field $\hat{X}^a$, is given by

$$\boldsymbol{J}_{\mathrm{T}}[\hat{\boldsymbol{X}}] \equiv {}^\star\hat{\boldsymbol{\jmath}}_{\mathrm{T}}[\hat{\boldsymbol{X}}], \tag{5.17}$$

where

$$\hat{\jmath}^a_{\mathrm{T}}[\hat{\boldsymbol{X}}] \equiv \hat{T}^{ab}\hat{X}_b = \frac{1}{4\pi}\left(\hat{F}^{ac}\hat{F}_{bc}\hat{X}^b - \frac{1}{4}\hat{X}^a\hat{F}^2\right). \tag{5.18}$$

Since this current has a divergence given by

$$\hat{\nabla}_a \hat{\jmath}^a_{\mathrm{T}}[\hat{\boldsymbol{X}}] = \hat{T}^{ab}\hat{\nabla}_{(a}\hat{X}_{b)}, \tag{5.19}$$

[2] Note that varying the Lagrangian with respect to $\hat{g}_{ab}$ is not in contradiction with our assumption that $\hat{g}_{ab}$ is non-dynamical in this section—$\hat{g}_{ab}$ does not satisfy any equation of motion obtained by varying the purely electromagnetic Lagrangian.

it is clear that $\hat{\jmath}^a_{\rm T}[\hat{\boldsymbol{X}}]$ is conserved when $\hat{X}^a$ is Killing.

We finally turn to the Noether current. To obtain an expression for this current, we consider the variation of the Lagrangian under the transformation generated by $\hat{\boldsymbol{\xi}} = (\hat{\boldsymbol{X}}, \hat{\lambda})$, where the vector potential transforms as in equation (5.2) and the variation of the metric under diffeomorphisms is $\delta_{\hat{\xi}} \hat{g}_{ab} = \pounds_{\hat{X}} \hat{g}_{ab}$. This yields[3]

$$\delta_{\hat{\xi}} \boldsymbol{L}_{\rm EM} = \pounds_{\hat{X}} \boldsymbol{L}_{\rm EM} = {\rm d}\boldsymbol{\eta}[\hat{\boldsymbol{\xi}}], \tag{5.20}$$

where the 3-form $\boldsymbol{\eta}[\hat{\boldsymbol{\xi}}]$ is given by

$$\boldsymbol{\eta}[\hat{\boldsymbol{\xi}}] = \hat{X} \cdot \boldsymbol{L}_{\rm EM}. \tag{5.21}$$

The *Noether current* is then defined by (see the appendix of [104])

$$\boldsymbol{J}_{\rm N}[\hat{\boldsymbol{\xi}}] \equiv \boldsymbol{\theta}_{\rm EM}[\hat{\boldsymbol{A}}; \delta_{\hat{\xi}} \hat{\boldsymbol{A}}] - \boldsymbol{\eta}[\hat{\boldsymbol{\xi}}] \equiv {}^\star \hat{\boldsymbol{\jmath}}_{\rm N}[\hat{\boldsymbol{\xi}}], \tag{5.22}$$

where

$$\hat{\jmath}^a_{\rm N}[\hat{\boldsymbol{\xi}}] = -\frac{1}{4\pi} \hat{F}^{ab} \left[\pounds_{\hat{X}} \hat{A}_b + \hat{\nabla}_b \hat{\lambda} \right] + \frac{1}{16\pi} \hat{X}^a \hat{F}^2. \tag{5.23}$$

Despite the fact that these three currents are clearly different, in the case where the vector field $\hat{X}^a$ is Killing, all these currents differ only by total derivatives and constant factors. It can be shown quite generally that the Noether and stress-energy currents are related by a total derivative [104]. For electromagnetic fields, we find by comparing the Noether and stress-energy current that

$$\boldsymbol{J}_{\rm N}[\hat{\boldsymbol{\xi}}] = -\boldsymbol{J}_{\rm T}[\hat{\boldsymbol{X}}] + {\rm d}\boldsymbol{Q}_{\rm N}[\hat{\boldsymbol{\xi}}], \tag{5.24}$$

where

$$\boldsymbol{Q}_{\rm N}[\hat{\boldsymbol{\xi}}] \equiv -\frac{1}{4\pi} {}^\star \hat{\boldsymbol{F}} \left(\hat{\boldsymbol{X}} \cdot \hat{\boldsymbol{A}} + \hat{\lambda} \right). \tag{5.25}$$

Comparing the canonical with the Noether current, one instead finds [after a lengthy but straightforward calculation starting with equation (5.14)] that

$$\boldsymbol{J}_{\rm C}[\hat{\boldsymbol{\xi}}] = 2\boldsymbol{J}_{\rm N}[\hat{\boldsymbol{\xi}}] + {\rm d}\boldsymbol{Q}_{\rm C}[\hat{\boldsymbol{\xi}}] + {}^\star \boldsymbol{K}_{\rm C}[\hat{\boldsymbol{X}}], \tag{5.26}$$

[3] Note that, when the vector field $\hat{X}^a$ is non-vanishing, $\delta_{\hat{\xi}} \boldsymbol{L}_{\rm EM}$ is a total derivative only when the non-dynamical metric in the Lagrangian also be varied.

where

$$\boldsymbol{Q}_{\mathrm{C}}[\hat{\boldsymbol{\xi}}] \equiv \frac{1}{4\pi}{}^{\star}\left[2\hat{\boldsymbol{X}} \wedge (\hat{\boldsymbol{A}} \cdot \hat{\boldsymbol{F}}) + \hat{\lambda}\hat{\boldsymbol{F}}\right], \tag{5.27}$$

$$K_{\mathrm{C}}^{a}[\hat{\boldsymbol{X}}] \equiv \frac{1}{2\pi}\left(2\hat{g}^{c[a}\hat{F}^{b]d} - \frac{1}{2}\hat{F}^{ab}\hat{g}^{cd}\right)\hat{A}_b\hat{\nabla}_{(c}\hat{X}_{d)}. \tag{5.28}$$

When $\hat{X}^a$ is a Killing vector field of the background spacetime, the Noether and canonical currents differ only by a total derivative of $\boldsymbol{Q}_{\mathrm{C}}[\hat{\boldsymbol{\xi}}]$ (up to a constant factor of two).

For any Killing vector field $\hat{X}^a$, these currents are all related by total derivatives, and the fact that the stress-energy current is conserved in this case directly shows that the other two currents are also conserved. From the discussion under equation (5.2), it follows that both the stress-energy and Noether current are invariant under gauge transformations, while the canonical current is invariant only up to boundary terms. Thus, we can use any of these currents to define a conserved quantity for electromagnetic fields associated with a Killing vector field of the background spacetime.[4] For example, if the background spacetime is stationary, possessing a timelike Killing field $\hat{t}^a$, then any of the above defined currents, setting $\hat{X}^a = \hat{t}^a$, defines a notion of "energy" when integrated over a Cauchy surface. Similarly, for an axisymmetric background, where there is an axial Killing field $\hat{X}^a = \hat{\phi}^a$, each of these currents defines a notion of "angular momentum". The conserved quantities defined using these currents will then differ by boundary terms on the Cauchy surface, either at a boundary at infinity, or some interior boundary, such as a black hole horizon.

As such, there are many different notions of "energy" and "angular momentum" on such backgrounds. Which of these currents is most appropriate depends, of course, on the problem at hand. The Noether current is the most natural current associated with a symmetry by Noether's theorem; moreover, as we will show, it is also the contribution to the Wald-Zoupas flux due to the electromagnetic fields in Einstein-Maxwell. On the other hand, the stress-energy current is typically used for calculations of energy and angular momentum flux, both in standard textbooks on electromagnetism in flat spacetimes [85, 105] and on fixed backgrounds [176] (in fact, problem 9.8 of [105] notes that the angular momentum flux depends on more than just the radiative electromagnetic fields). Furthermore, the stress-energy current is useful for computations of "self-force" effects on charged sources due to electromagnetic radiation; see, for instance [138, 39].

[4] Of course, one is free to define other conserved currents by simply adding exact 2-forms (that is, boundary terms) to the three currents we have defined.

There are also several situations in which the canonical current is most useful. The canonical currents arise naturally in the Hamiltonian formulation of electromagnetism, as the symplectic current provides a natural symplectic form on phase space. These currents also arise in the formulation of the first law of black hole mechanics [104, 133]. Moreover, these currents are useful in analyzing the stability of solutions in classical field theories: by general arguments, the positivity of the canonical energy (relative to a timelike Killing field of the background) is directly related to the stability of the background black hole to perturbations [96, 134]. For axisymmetric electromagnetic fields on a stationary (but not static) and axisymmetric black hole spacetime in GR, it was shown in [135] that the energy evaluated on a Cauchy surface defined by the canonical current (which, in this case, also equals the one defined by Noether current) is positive, whereas the energy given by the stress-energy tensor can be made negative. The canonical energy is also useful to account for the "second-order" self-force effects of small test bodies in black hole spacetimes [150]. Similarly, the symplectic current is useful for deriving conserved currents associated with symmetries of the equations of motion; such symmetries need not arise from the action of a diffeomorphism or gauge transformation, and do not necessarily possess currents associated with the stress-energy tensor [81, 82].

5.1.2 Asymptotic structure

We now review the asymptotic structure of electromagnetism on a fixed background, using the usual conformal completion definition of null infinity ($\mathscr{I}$) with conformal factor Ω (for a review, see [77]). Here, the physical spacetime $(\hat{M}, \hat{g}_{ab})$ is replaced by an unphysical spacetime (M, g_{ab}), where $g_{ab} = \Omega^2 \hat{g}_{ab}$, for a given scalar field Ω. Null infinity is then the subset of M where $\Omega = 0$. Moreover, this conformal factor is assumed to satisfy

$$\nabla_a \Omega|_{\mathscr{I}} \neq 0. \tag{5.29}$$

For definiteness, we will only consider *future* null infinity—depending on conventions, some of our formulae will acquire an additional sign when instead considering past null infinity.

Given the relationship $g_{ab} = \Omega^2 \hat{g}_{ab}$, the conversion between the metrics and volume elements in the physical and unphysical spacetimes is given by

$$\hat{g}_{ab} = \Omega^{-2} g_{ab}, \qquad \hat{g}^{ab} = \Omega^2 g^{ab}, \qquad \hat{\epsilon}_{abcd} = \Omega^{-4} \epsilon_{abcd}. \tag{5.30}$$

While the unphysical metric $\hat{g}_{ab}$ does not have a smooth limit to $\mathscr{I}$, g_{ab} does. Let $n_a \equiv \nabla_a \Omega$. It can be shown that the conformal factor Ω can always be chosen so that the *Bondi condition*,

$$\nabla_a n_b \mathrel{\widehat{=}} 0, \tag{5.31}$$

is satisfied, where "$\mathrel{\widehat{=}}$" denotes equality on $\mathscr{I}$. Furthermore, with this choice, one can show that

$$n_a n^a = O(\Omega^2). \tag{5.32}$$

We will work with this choice of conformal factor throughout. Using this conformal factor, one finds that the conformal factor Ω in a neighborhood of $\mathscr{I}$, as well as the unphysical metric at $\mathscr{I}$, $g_{ab}|_{\mathscr{I}}$, are universal; that is, they are independent of the choice of physical metric $\hat{g}_{ab}$ [78, 178]. This is essentially because *all* asymptotically flat spacetimes can have their neighborhoods of $\mathscr{I}$ mapped into one another, preserving $g_{ab}|_{\mathscr{I}}$ and Ω (see Appendix A of [73] for further details of this argument).

Let q_{ab} denote the pullback of the unphysical metric g_{ab} to $\mathscr{I}$. From equations (5.31) and (5.32), it follows that $q_{ab} n^b \mathrel{\widehat{=}} 0$ and $\pounds_n q_{ab} \mathrel{\widehat{=}} 0$. Thus, q_{ab} defines a degenerate metric on $\mathscr{I}$ and a Riemannian metric on the space of null generators (diffeomorphic to $\mathbb{S}^2$) of $\mathscr{I}$.

For our computations, it will be convenient to define some additional structure on $\mathscr{I}$ as follows. Let u be a function on $\mathscr{I}$ such that $n^a \nabla_a u \mathrel{\widehat{=}} 1$; that is, u is a coordinate along the null generators of $\mathscr{I}$ with $n^a \mathrel{\widehat{=}} (\partial_u)^a$. Consider the foliation of $\mathscr{I}$ by a family of cross-sections of constant u. The pullback of q_{ab} to any such cross-section S defines a Riemannian metric on S. For such a choice of foliation, there is a unique *auxiliary normal* vector field l^a at $\mathscr{I}$ such that[5]

$$l^a l_a \mathrel{\widehat{=}} 0, \qquad l^a n_a \mathrel{\widehat{=}} -1, \qquad q_{ab} l^b \mathrel{\widehat{=}} 0. \tag{5.33}$$

This choice of auxiliary normal is, moreover, parallel-transported along n^a, at least on $\mathscr{I}$:

$$n^b \nabla_b l^a \mathrel{\widehat{=}} 0. \tag{5.34}$$

[5] All of our results can be obtained without choosing a foliation of $\mathscr{I}$ and the corresponding auxiliary normal l^a, but some intermediate computations become more cumbersome; see [77, 18].

In terms of this auxiliary normal, we also have

$$q_{ab} \mathrel{\hat{=}} g_{ab} + 2n_{(a}l_{b)}, \qquad q^{ab} \mathrel{\hat{=}} g^{ab} + 2n^{(a}l^{b)}. \tag{5.35}$$

where q^{ab} is the "inverse metric" on the chosen foliation relative to l^a. For any v_a satisfying $n^a v_a \mathrel{\hat{=}} l^a v_a \mathrel{\hat{=}} 0$ on $\mathscr{I}$, we define the derivative $\mathscr{D}_a$ on the cross-sections by

$$\mathscr{D}_a v_b \equiv q_a{}^c q_b{}^d \nabla_c v_d. \tag{5.36}$$

It is easily verified that $\mathscr{D}_a q_{bc} \mathrel{\hat{=}} 0$; that is, $\mathscr{D}_a$ is the metric-compatible covariant derivative on cross-sections of $\mathscr{I}$.

Let $\boldsymbol{\epsilon}_3$ (denoted with indices by ϵ_{abc}) be the volume element on $\mathscr{I}$, and let $\boldsymbol{\epsilon}_2$ (denoted with indices by ϵ_{ab}) be the area element on the cross-sections of $\mathscr{I}$ in our choice of foliation. We define these volume elements by

$$\epsilon_{abc} \equiv l^d \epsilon_{dabc}, \qquad \epsilon_{ab} \equiv -n^c \epsilon_{cab}. \tag{5.37}$$

These are the orientations of $\boldsymbol{\epsilon}_3$ and $\boldsymbol{\epsilon}_2$ that are used by [178]. In our choice of foliation, we also have $\boldsymbol{\epsilon}_3 = -du \wedge \boldsymbol{\epsilon}_2$.

5.1.2.1 Asymptotic electromagnetic fields

We now consider the asymptotic behavior of the electromagnetic fields. The Maxwell equations are conformally invariant, assuming that the unphysical Faraday tensor is given by $F_{ab} = \hat{F}_{ab}$, and we assume that F_{ab} extends smoothly to $\mathscr{I}$. For the vector potential, this implies that there exists a gauge in which $A_a = \hat{A}_a$ is also smooth at $\mathscr{I}$.[6] Moreover, without loss of generality—that is, for all solutions of the Maxwell equations where F_{ab} is smooth at $\mathscr{I}$—we can further impose the condition of *outgoing radiation gauge*:

$$n^a A_a \mathrel{\hat{=}} 0. \tag{5.38}$$

The argument is similar to the one used for imposing the Bondi condition (see for instance, Sec. 11.1 of [176]): let A_a be a vector potential so that $n^a A_a \mathrel{\hat{\neq}} 0$, and consider another vector potential

[6] Generically, if we impose some gauge condition on $\hat{A}_a$ in the physical spacetime, for example Lorenz gauge, then $A_a = \hat{A}_a$ is not guaranteed to be smooth at $\mathscr{I}$ in the chosen gauge; see, for example, the case of Kerr-Newman spacetime in section 5.3.2.2.

$A'_a = A_a + \nabla_a \lambda$. Now choose λ to be a solution of

$$\pounds_n \lambda \mathrel{\hat{=}} -n^a A_a. \tag{5.39}$$

Since this is an ordinary differential equation along the generators of $\mathscr{I}$, solutions to this equation always exist. With this choice of λ, we find $n^a A'_a \mathrel{\hat{=}} 0$. Henceforth, we will assume that this choice has been made for the vector potential. In summary, we have that

$$\hat{A}_a = A_a, \qquad \hat{F}_{ab} = F_{ab}, \tag{5.40}$$

are all smooth at $\mathscr{I}$ along with the condition in equation (5.38).

We now consider the "electric field" $F_{ab}n^b$ at $\mathscr{I}$. Two important quantities can be derived from this field: first, we consider its pullback $\mathcal{E}_a$:

$$\mathcal{E}_a \equiv \underleftarrow{F_{ab} n^b} = q_a{}^c F_{cb} n^b = -\pounds_n \underleftarrow{A_a}, \tag{5.41}$$

with the under arrow indicating the pullback to $\mathscr{I}$. The radiative degrees of freedom in the electromagnetic field are contained in $\mathcal{E}_a$ (or, equivalently, $\underleftarrow{A}_a$). The other piece of $F_{ab}n^b$, which contains non-radiative (Coulombic) information at $\mathscr{I}$, is given by $\mathrm{Re}[\varphi_1]$, defined by[7]

$$\mathrm{Re}[\varphi_1] \equiv \frac{1}{2} F_{ab} l^a n^b. \tag{5.42}$$

The Maxwell equations imply that, on $\mathscr{I}$, these two fields are related in the following way:

$$2\pounds_n \mathrm{Re}[\varphi_1] \mathrel{\hat{=}} q^{ab} \mathscr{D}_a \mathcal{E}_b. \tag{5.43}$$

5.1.2.2 Asymptotic symmetries

In this section, we determine asymptotic symmetries of electromagnetism on a fixed background from the asymptotic conditions on the gravitational and electromagnetic fields at null infinity. We first focus on the asymptotic symmetries of the gravitational field, before we include the symmetry transformations of the vector potential. Similar arguments for vacuum general relativity were also presented in [73].

[7] The notation "$\mathrm{Re}[\varphi_1]$" comes from Newman-Penrose notation [121]. Similarly, the quantity $\mathcal{E}_a$ corresponds to the real and imaginary parts of φ_2 in Newman-Penrose notation.

Given a vector field $\hat{X}^a = X^a$ that generates an infinitesimal diffeomorphism $\pounds_X \hat{g}_{ab}$ in the physical spacetime, we must first determine the conditions on X^a for it to be an asymptotic symmetry vector field. The vector field X^a needs to extend smoothly to $\mathscr{I}$ to preserve the smooth differential structure there, and the unphysical metric perturbation determined by the infinitesimal diffeomorphisms generated by X^a needs to preserve the asymptotic flatness condition. To make this concrete, consider any physical metric perturbation $\delta_X \hat{g}_{ab} = \pounds_{\hat{X}} \hat{g}_{ab}$ generated by a diffeomorphism. The corresponding unphysical metric perturbation is given by

$$\delta_X g_{ab} = \Omega^2 \pounds_X \hat{g}_{ab} = \pounds_X g_{ab} - 2\Omega^{-1} n_c X^c g_{ab}, \tag{5.44}$$

using the fact that Ω can be chosen to be universal, as mentioned in section 5.1.2. Since $\delta_X g_{ab}$ has to be smooth at $\mathscr{I}$, we can immediately conclude that $n_a X^a \mathrel{\hat{=}} 0$. In other words, X^a is tangent to $\mathscr{I}$. Defining the function $\alpha_{(X)} \equiv \Omega^{-1} n_a X^a$, which extends smoothly to $\mathscr{I}$, we can write the above equation as

$$\delta_X g_{ab} = \pounds_X g_{ab} - 2\alpha_{(X)} g_{ab}. \tag{5.45}$$

The perturbation $\delta_X g_{ab}$ must vanish at $\mathscr{I}$, as $g_{ab}|_{\mathscr{I}}$ is universal:

$$\delta_X g_{ab} \mathrel{\hat{=}} 0, \tag{5.46}$$

which implies that

$$\pounds_X g_{ab} \mathrel{\hat{=}} 2\alpha_{(X)} g_{ab}. \tag{5.47}$$

Similarly, assuming that the diffeomorphism associated with X^a preserves the Bondi condition in equation (5.31), we require that

$$n^a n^b \delta_X g_{ab} = O(\Omega^2). \tag{5.48}$$

Furthermore, contracting equation (5.45) with n^b gives

$$n^b \delta_X g_{ab} = n^b \nabla_b X_a - X^b \nabla_b n_a - \alpha_{(X)} n_a + \Omega \nabla_a \alpha_{(X)}, \tag{5.49}$$

where we have used that the twist of n_a vanishes, since n_a is the gradient of the conformal factor Ω. Since the left-hand side must vanish at $\mathscr{I}$, we have

$$n^b \delta_X g_{ab} \mathrel{\hat{=}} 0 \implies \pounds_X n^a \mathrel{\hat{=}} -\alpha_{(X)} n^a. \tag{5.50}$$

Contracting equation (5.49) once more with n^a, we find that

$$n^a n^b \delta_X g_{ab} = O(\Omega^2) \implies \pounds_n \alpha_{(X)} \hat{=} 0, \tag{5.51}$$

where we used $n_a n^a = O(\Omega^2)$ [see equation (5.32), which followed directly from the Bondi condition in equation (5.31)]. Finally, taking the pullback of equation (5.47) to $\mathscr{I}$, we find

$$\pounds_X q_{ab} \hat{=} 2\alpha_{(X)} q_{ab}. \tag{5.52}$$

Hence, the asymptotic symmetries on $\mathscr{I}$ are generated by vector fields X^a tangent to $\mathscr{I}$ satisfying

$$\pounds_X n^a \hat{=} -\alpha_{(X)} n^a, \tag{5.53a}$$

$$\pounds_X q_{ab} \hat{=} 2\alpha_{(X)} q_{ab}, \tag{5.53b}$$

where the function $\alpha_{(X)}$ is smooth and satisfies $\pounds_n \alpha_{(X)} \hat{=} 0$ on $\mathscr{I}$. These are the standard conditions defining the BMS algebra $\mathfrak{b}$ [77, 18]. When working solely on $\mathscr{I}$, the function $\alpha_{(X)}$ can be interpreted as the infinitesimal conformal transformation of q_{ab} induced by $X^a|_{\mathscr{I}}$. If X^a is given in a neighborhood of $\mathscr{I}$, $\alpha_{(X)}$ can also be computed using

$$\alpha_{(X)} \hat{=} \Omega^{-1} n_a X^a \hat{=} \frac{1}{4} \nabla_a X^a, \tag{5.54}$$

where the second equality follows from the fact that $g^{ab}\delta_X g_{ab} \hat{=} 0$.

To make these conditions more concrete, let us use the foliation of $\mathscr{I}$ by the parameter u defined above. Then any BMS vector field can be written as

$$X^a \hat{=} \beta n^a + Y^a, \tag{5.55}$$

and the conditions in equations (5.53) and (5.54) yield

$$\beta \hat{=} f + \frac{1}{2}(u - u_0)\mathscr{D}_a Y^a, \tag{5.56}$$

where

$$\pounds_n f \hat{=} \pounds_n Y^a \hat{=} 0, \qquad 2\mathscr{D}_{(a} Y_{b)} \hat{=} q_{ab} \mathscr{D}_c Y^c. \tag{5.57}$$

Here, Y^a is tangent to the cross-sections of constant u of $\mathscr{I}$ and $u = u_0$ is some choice of an "origin" cross-section. The function $\alpha_{(X)}$ in (5.53) is then given by $\frac{1}{2}\mathscr{D}_a Y^a$ in this representation. Thus, any

BMS vector field is characterized by a smooth function f and a smooth conformal Killing field Y^a on $\mathbb{S}^2$. The function f represents the infinite-dimensional subalgebra of *supertranslations*, while the conformal Killing field Y^a represents a *Lorentz* subalgebra of the full BMS Lie algebra.

Given a *fixed* BMS vector field X^a, its representation in terms of a supertranslation f and a Lorentz vector field Y^a depends on the choice of foliation given by cross-sections of constant u. Let $u' = u + F$ with $\pounds_n F \mathrel{\hat{=}} 0$ be another choice of affine parameter along n^a, and let f' and Y'^a be representatives of X^a in the new choice of foliation given by surfaces of constant u'. Then it is straightforward to verify that

$$f' \mathrel{\hat{=}} f + \pounds_Y F, \qquad Y'^a \mathrel{\hat{=}} Y^a, \tag{5.58}$$

where in the second equation we are identifying vector fields tangent to two different cross-sections of $\mathscr{I}$. Therefore, the notion of a pure supertranslation ($Y^a \mathrel{\hat{=}} 0$) is well-defined independently of the choice of foliation, but a "pure Lorentz" transformation ($f = 0$) is not. This is ultimately related to the fact that the BMS algebra is a semidirect sum of the Lorentz algebra with the Lie ideal of supertranslations.

Now consider a similar analysis of the transformations of the vector potential under a symmetry $\boldsymbol{\xi} = (\boldsymbol{X}, \lambda)$, where X^a is a BMS vector field and $\lambda = \hat{\lambda}$. The transformation in equation (5.2) needs to preserve the asymptotic conditions of the vector potential. Since A_a is smooth at $\mathscr{I}$, λ extends smoothly to $\mathscr{I}$ as well. To preserve the outgoing gauge condition imposed on the vector potential [equation (5.38)], one requires that $n^a \delta_\xi A_a \mathrel{\hat{=}} 0$: using equation (5.53a), one finds that

$$0 \mathrel{\hat{=}} n^a \pounds_X A_a + \pounds_n \lambda \mathrel{\hat{=}} \pounds_X (n^a A_a) + \alpha_{(X)} n^a A_a + \pounds_n \lambda, \tag{5.59}$$

and so $n^a A_a \mathrel{\hat{=}} 0$ implies that

$$\pounds_n \lambda \mathrel{\hat{=}} 0. \tag{5.60}$$

Thus, the asymptotic symmetries of this theory at $\mathscr{I}$ are given by $\boldsymbol{\xi} = (\boldsymbol{X}, \lambda)$, where X^a is a BMS vector field and λ is any smooth function on $\mathbb{S}^2$, the space of null generators of $\mathscr{I}$.

5.1.3 | Asymptotic fluxes

Using the results of the previous section, we now evaluate the fluxes through null infinity defined by the canonical, Noether, and stress-energy currents for any asymptotic symmetry $\boldsymbol{\xi} = (\boldsymbol{X}, \lambda)$ as described above. Note that in this context the vector field $\hat{X}^a = X^a$ need not be a Killing vector field inside the physical spacetime but is required to be a BMS vector field on $\mathscr{I}$.

The relevant fluxes are determined by integrating the pullbacks of the 3-form currents $\boldsymbol{J}$, defined in terms of some vector current $\hat{\jmath}^a$ by $\boldsymbol{J} = {}^\star\hat{\boldsymbol{\jmath}}$. With our convention in equation (5.37) for $\boldsymbol{\epsilon}_3$, the pullback of $\boldsymbol{J}$ is given by $-\Omega^{-4} n_a \hat{\jmath}^a\, \boldsymbol{\epsilon}_3$. The flux of the canonical current is given by

$$\begin{aligned}\mathcal{F}_{\rm C}[\boldsymbol{\xi}; \Delta\mathscr{I}] &\equiv \int_{\Delta\mathscr{I}} \boldsymbol{J}_{\rm C}[\boldsymbol{\xi}] = -\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3\ \Omega^{-4} n_a \hat{\jmath}^a_{\rm C}[\boldsymbol{\xi}] \\ &= -\frac{1}{4\pi}\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3\ q^{ab}\left[\mathcal{E}_a(\pounds_X A_b + \mathscr{D}_b\lambda) - A_a \pounds_X \mathcal{E}_b - \frac{1}{2}\mathcal{E}_a A_b \mathscr{D}_c Y^c\right],\end{aligned} \tag{5.61}$$

where Y^a is the "pure Lorentz part" of X^a and we have used that $\pounds_X n^a \mathrel{\hat=} -\frac{1}{2}(\mathscr{D}_b Y^b) n^a$ [see equation (5.53) and the text below equation (5.57)]. The flux of the Noether current, similarly, is given by

$$\mathcal{F}_{\rm N}[\boldsymbol{\xi}; \Delta\mathscr{I}] \equiv \int_{\Delta\mathscr{I}} \boldsymbol{J}_{\rm N}[\boldsymbol{\xi}] = -\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3\ \Omega^{-4} n_a \hat{\jmath}^a_{\rm N}[\boldsymbol{\xi}] = -\frac{1}{4\pi}\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3\ q^{ab}\mathcal{E}_a(\pounds_X A_b + \mathscr{D}_b\lambda), \tag{5.62}$$

where we have used that $\pounds_n \lambda \mathrel{\hat=} 0$ [see equation (5.60)]. The term proportional to F^2 in equation (5.23) does not contribute to the flux through $\mathscr{I}$ because $X^a n_a \mathrel{\hat=} 0$. Finally, the flux of the stress-energy current is given by

$$\mathcal{F}_{\rm T}[\boldsymbol{\xi}; \Delta\mathscr{I}] \equiv \int_{\Delta\mathscr{I}} \boldsymbol{J}_{\rm T}[\boldsymbol{\xi}] = -\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3\ T_{ab} n^a X^b = -\frac{1}{4\pi}\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3\ \mathcal{E}_a\left(q^{ab} F_{bc} X^c + 2\,\mathrm{Re}[\varphi_1] Y^a\right), \tag{5.63}$$

where $T_{ab} \equiv \Omega^{-2}\hat{T}_{ab}$ is smooth at $\mathscr{I}$.

From the above expressions, it is apparent that all of these fluxes vanish in the absence of electromagnetic radiation, that is, when $\mathcal{E}_a = 0$. Furthermore, the fluxes defined by the Noether and canonical currents depend only on the radiative degrees of freedom $\underleftarrow{A}_a$ at null infinity. However, the stress-energy current flux also depends on the Coulombic part $\mathrm{Re}[\varphi_1]$, as emphasized before in [19, 20]. For supertranslations $X^a \propto n^a$, this Coulombic term does not contribute to the flux, since in this case $Y^a = 0$. However, the stress-energy current flux associated with asymptotic Lorentz symmetries, for example the flux of angular momentum, *cannot* be computed from radiative degrees of freedom.

Note that, since any BMS vector field satisfies $\Omega^2 \pounds_X \hat{g}_{ab} \mathrel{\hat=} 0$ (see the discussion in section 5.1.2.2), the term $\boldsymbol{K}_{\rm C}[\boldsymbol{X}]$ in equation (5.26) vanishes at null infinity. Thus, from equation (5.26), we have that, on $\mathscr{I}$,

$$\boldsymbol{J}_{\mathrm{N}}[\boldsymbol{\xi}] \mathrel{\hat{=}} \frac{1}{2}\left\{\boldsymbol{J}_{\mathrm{C}}[\boldsymbol{\xi}] - \mathrm{d}\boldsymbol{Q}_{\mathrm{C}}[\boldsymbol{\xi}]\right\}, \qquad \boldsymbol{J}_{\mathrm{T}}[\boldsymbol{\xi}] \mathrel{\hat{=}} -\boldsymbol{J}_{\mathrm{N}}[\boldsymbol{\xi}] + \mathrm{d}\boldsymbol{Q}_{\mathrm{N}}[\boldsymbol{\xi}]. \tag{5.64}$$

That is, all three currents evaluated on $\mathscr{I}$ differ by exact 3-forms, even when the vector field X^a is not Killing but an element of the BMS algebra. Therefore, the fluxes of these currents on $\mathscr{I}$ are related to each other by boundary terms on the cross-sections S_2 and S_1 bounding the region $\Delta\mathscr{I}$ (with S_2 in the future of S_1).

Let us compare the fluxes on $\mathscr{I}$ in more detail. Consider, first, the relation between the flux of the Noether and canonical current. This satisfies

$$\mathcal{F}_{\mathrm{N}}[\boldsymbol{\xi};\Delta\mathscr{I}] \equiv \int_{\Delta\mathscr{I}} \boldsymbol{J}_{\mathrm{N}}[\boldsymbol{\xi}] = \frac{1}{2}\mathcal{F}_{\mathrm{C}}[\boldsymbol{\xi};\Delta\mathscr{I}] + \frac{1}{2}\left\{\int_{S_2} \boldsymbol{Q}_{\mathrm{C}}[\boldsymbol{\xi}] - \int_{S_1} \boldsymbol{Q}_{\mathrm{C}}[\boldsymbol{\xi}]\right\}, \tag{5.65}$$

with the boundary term

$$\int_S \boldsymbol{Q}_{\mathrm{C}}[\xi] = -\frac{1}{4\pi}\int_S \boldsymbol{\epsilon}_2\left(\beta\mathcal{E}^a A_a - 2\lambda\ \mathrm{Re}[\varphi_1]\right), \tag{5.66}$$

where β is as given in equation (5.55). This expression is rather strange on first inspection, since both $\mathcal{F}_{\mathrm{C}}[\boldsymbol{\xi};\Delta\mathscr{I}]$ and $\mathcal{F}_{\mathrm{N}}[\boldsymbol{\xi};\Delta\mathscr{I}]$ contain only radiative information by equations (5.61) and (5.62), respectively, and yet their difference appears to be a boundary term that contains non-radiative information, in the form of $\lambda\,\mathrm{Re}[\varphi_1]$. This is somewhat misleading, since using equation (5.43) and $\pounds_n\lambda \mathrel{\hat{=}} 0$, this Coulombic contribution can be rewritten in terms of purely radiative degrees of freedom as

$$\frac{1}{4\pi}\int_{S_2}\boldsymbol{\epsilon}_2\,2\lambda\,\mathrm{Re}[\varphi_1] - \frac{1}{4\pi}\int_{S_1}\boldsymbol{\epsilon}_2\,2\lambda\,\mathrm{Re}[\varphi_1] = \frac{1}{4\pi}\int_{\Delta\mathscr{I}}\boldsymbol{\epsilon}_3\,q^{ab}\mathcal{E}_a\mathscr{D}_b\lambda. \tag{5.67}$$

Next, consider the relation between the flux of the stress-energy and Noether current:

$$\mathcal{F}_{\mathrm{T}}[\boldsymbol{\xi};\Delta\mathscr{I}] = -\mathcal{F}_{\mathrm{N}}[\boldsymbol{\xi};\Delta\mathscr{I}] - \left\{\int_{S_2}\boldsymbol{Q}_{\mathrm{N}}[\boldsymbol{\xi}] - \int_{S_1}\boldsymbol{Q}_{\mathrm{N}}[\boldsymbol{\xi}]\right\}, \tag{5.68}$$

where

$$\int_S \boldsymbol{Q}_{\mathrm{N}}[\boldsymbol{\xi}] = -\frac{1}{2\pi}\int_S \boldsymbol{\epsilon}_2\,\mathrm{Re}[\varphi_1]\left(Y^a A_a + \lambda\right). \tag{5.69}$$

Unsurprisingly, as there is non-radiative information in $\mathcal{F}_{\mathrm{T}}[\boldsymbol{\xi};\Delta\mathscr{I}]$ but not in $\mathcal{F}_{\mathrm{N}}[\boldsymbol{\xi};\Delta\mathscr{I}]$, the boundary term contains non-radiative information.

Finally, let us consider the fluxes through all of $\mathscr{I}$. The natural boundary conditions for the electromagnetic field in the limit $u \to \pm\infty$ are

$$\mathcal{E}_a = O(1/|u|^{1+\epsilon}), \qquad \underleftarrow{A_a} = O(1). \tag{5.70}$$

These conditions ensure that the symplectic form obtained by integrating the symplectic current over *all* of $\mathscr{I}$ is finite. Given that β grows at most linearly in u and Y^a and λ are independent of u (see section 5.1.2.2), we find that the fluxes differ by

$$\mathcal{F}_\mathrm{N}[\boldsymbol{\xi}; \mathscr{I}] = \frac{1}{2}\mathcal{F}_\mathrm{C}[\boldsymbol{\xi}; \mathscr{I}] + \frac{1}{2}\left\{\mathcal{Q}_\mathrm{C}[\boldsymbol{\xi}; S_\infty] - \mathcal{Q}_\mathrm{C}[\boldsymbol{\xi}; S_{-\infty}]\right\}, \tag{5.71}$$

$$\mathcal{F}_\mathrm{T}[\boldsymbol{\xi}; \mathscr{I}] = -\mathcal{F}_\mathrm{N}[\boldsymbol{\xi}; \mathscr{I}] - \left\{\mathcal{Q}_\mathrm{N}[\boldsymbol{\xi}; S_\infty] - \mathcal{Q}_\mathrm{N}[\boldsymbol{\xi}; S_{-\infty}]\right\}, \tag{5.72}$$

where S_∞ and $S_{-\infty}$ are the spheres at $u = \pm\infty$, respectively, and

$$\mathcal{Q}_\mathrm{C}[\boldsymbol{\xi}; S] \equiv \frac{1}{2\pi}\int_S \boldsymbol{\epsilon}_2 \, \lambda \, \mathrm{Re}[\varphi_1], \tag{5.73}$$

$$\mathcal{Q}_\mathrm{N}[\boldsymbol{\xi}; S] \equiv -\frac{1}{2\pi}\int_S \boldsymbol{\epsilon}_2 \; \mathrm{Re}[\varphi_1](Y^a A_a + \lambda). \tag{5.74}$$

As discussed below equation (5.66), the difference between the canonical and Noether fluxes can also be expressed purely in terms of the radiative degrees of freedom. However, the difference between the Noether and stress-energy fluxes depends on the Coulombic degrees of freedom, even when computed over all of $\mathscr{I}$, except when $Y^a = 0$ and $\lambda = 0$ (a pure supertranslation).

We stress once more that none of these fluxes can be written as the difference of charges evaluated on cross-sections of null infinity. Thus, on a non-dynamical background spacetime, none of these fluxes can be considered as the change of energy or angular momentum at a particular "time" (a cross-section of null infinity), and there is no obvious criterion to decide which of these currents defines the flux of energy or angular momentum.

5.2 Review of Einstein-Maxwell theory

In this section, we review basic properties of Einstein-Maxwell theory, first covering the symplectic structure of the theory in section 5.2.1, and then turning to the behavior of the asymptotic fields in section 5.2.2. We conclude this section with a handful of theorems concerning stationary solutions in this theory in section 5.2.3.

5.2.1 Symplectic structure

Following [44], the Lagrangian for Einstein-Maxwell theory is given by

$$\boldsymbol{L} = \frac{1}{16\pi}\left(\hat{R} - \hat{F}^2\right)\hat{\boldsymbol{\epsilon}}_4. \tag{5.75}$$

As in the case of electromagnetism on a fixed background, our analysis is unaffected by adding additional matter sources of either compact support or sufficiently fast falloff at null infinity.

A variation of this Lagrangian with respect to the dynamical fields $\hat{\boldsymbol{\Phi}} = (\hat{\boldsymbol{g}}, \hat{\boldsymbol{A}})$ gives (raising and lowering with the background physical metric)

$$\delta\boldsymbol{L} = \left[-\frac{1}{16\pi}(\hat{G}^{ab} - 8\pi\hat{T}^{ab})\delta\hat{g}_{ab} + \frac{1}{4\pi}\hat{\nabla}_b\hat{F}^{ba}\delta\hat{A}_a\right]\hat{\boldsymbol{\epsilon}}_4 + \mathrm{d}\boldsymbol{\theta}[\hat{\boldsymbol{\Phi}};\delta\hat{\boldsymbol{\Phi}}], \tag{5.76}$$

where $\hat{G}_{ab}$ is the Einstein tensor of $\hat{g}_{ab}$ and the stress-energy tensor $\hat{T}_{ab}$ is the same as in equation (5.16), except that the spacetime metric is now also dynamical. The variations with respect to the dynamical fields $\hat{g}_{ab}$ and $\hat{A}_a$ give the Einstein equations and Maxwell equations, respectively:

$$\hat{G}_{ab} = 8\pi\hat{T}_{ab}, \qquad \hat{\nabla}_b\hat{F}^{ba} = 0. \tag{5.77}$$

The symplectic potential $\boldsymbol{\theta}[\hat{\boldsymbol{\Phi}};\delta\hat{\boldsymbol{\Phi}}]$ is given by

$$\boldsymbol{\theta}[\hat{\boldsymbol{\Phi}};\delta\hat{\boldsymbol{\Phi}}] = {}^\star\hat{\boldsymbol{v}}[\hat{\boldsymbol{\Phi}};\delta\hat{\boldsymbol{\Phi}}], \tag{5.78}$$

where

$$\hat{v}^a[\hat{\boldsymbol{\Phi}};\delta\hat{\boldsymbol{\Phi}}] \equiv \frac{1}{8\pi}\left(\hat{g}^{a[b}\hat{g}^{c]d}\hat{\nabla}_c\delta\hat{g}_{bd} - 2\hat{F}^{ab}\delta\hat{A}_b\right), \tag{5.79}$$

where the second term is the symplectic potential of electromagnetism from equation (5.10). The symplectic current, as before, is defined by

$$\boldsymbol{\omega}[\hat{\boldsymbol{\Phi}};\delta_1\hat{\boldsymbol{\Phi}},\delta_2\hat{\boldsymbol{\Phi}}] \equiv \delta_1\boldsymbol{\theta}[\hat{\boldsymbol{\Phi}};\delta_2\hat{\boldsymbol{\Phi}}] - \delta_2\boldsymbol{\theta}[\hat{\boldsymbol{\Phi}};\delta_1\hat{\boldsymbol{\Phi}}], \tag{5.80}$$

and is given by the sum of three terms [see equation (3.12) of [44]][8]

$$\boldsymbol{\omega}[\hat{\boldsymbol{\Phi}};\delta_1\hat{\boldsymbol{\Phi}},\delta_2\hat{\boldsymbol{\Phi}}] \equiv {}^\star\left\{\hat{\boldsymbol{w}}_{\mathrm{GR}}[\hat{\boldsymbol{g}};\delta_1\hat{\boldsymbol{g}},\delta_2\hat{\boldsymbol{g}}] + \hat{\boldsymbol{w}}_{\mathrm{EM}}[\hat{\boldsymbol{g}};\delta_1\hat{\boldsymbol{A}},\delta_2\hat{\boldsymbol{A}}] + \hat{\boldsymbol{w}}_{\times}[\hat{\boldsymbol{\Phi}};\delta_1\hat{\boldsymbol{\Phi}},\delta_2\hat{\boldsymbol{\Phi}}]\right\}. \tag{5.81}$$

[8] Note that our expressions in equations (5.84) and (5.85) differ in appearance from the ones in equation (3.12) of [44] only because [44] uses the perturbed quantity $\delta\hat{F}^{ab}$, while we prefer to use $\delta\hat{F}_{ab}$.

The first term on the right-hand side of equation (5.81) is the same as the symplectic current for vacuum general relativity [see equations (41) and (42) of [178]]:

$$\hat{w}^a_{\rm GR}[\hat{\boldsymbol{g}};\delta_1\hat{\boldsymbol{g}},\delta_2\hat{\boldsymbol{g}}] = \frac{1}{16\pi}\hat{P}^{abcdef}\left[\delta_2\hat{g}_{bc}\hat{\nabla}_d\delta_1\hat{g}_{ef} - (1\leftrightarrow 2)\right], \tag{5.82}$$

with

$$\hat{P}^{abcdef} = \hat{g}^{ae}\hat{g}^{fb}\hat{g}^{cd} - \frac{1}{2}\hat{g}^{ad}\hat{g}^{be}\hat{g}^{fc} - \frac{1}{2}\hat{g}^{ab}\hat{g}^{cd}\hat{g}^{ef} - \frac{1}{2}\hat{g}^{bc}\hat{g}^{ae}\hat{g}^{fd} + \frac{1}{2}\hat{g}^{bc}\hat{g}^{ad}\hat{g}^{ef}. \tag{5.83}$$

Similarly, the second term is the symplectic current of electromagnetism from equation (5.12):

$$\hat{w}^a_{\rm EM}[\hat{\boldsymbol{g}};\delta_1\hat{\boldsymbol{A}},\delta_2\hat{\boldsymbol{A}}] = -\frac{1}{4\pi}\hat{g}^{ac}\hat{g}^{bd}\left[\delta_1\hat{F}_{cd}\delta_2\hat{A}_b - (1\leftrightarrow 2)\right], \tag{5.84}$$

while the third "cross-term" is given by

$$\hat{w}^a_\times[\hat{\boldsymbol{\Phi}};\delta_1\hat{\boldsymbol{\Phi}},\delta_2\hat{\boldsymbol{\Phi}}] = -\frac{1}{4\pi}\left(2\hat{g}^{c[a}\hat{F}^{b]d} + \frac{1}{2}\hat{F}^{ab}\hat{g}^{cd}\right)\delta_2\hat{A}_b\delta_1\hat{g}_{cd} - (1\leftrightarrow 2). \tag{5.85}$$

This cross-term is unimportant for our analysis, as it vanishes in the limit to $\mathscr{I}$ for asymptotically flat perturbations, as we will show in the next section.

5.2.2 | Asymptotic structure

We now review the asymptotic behavior of Einstein-Maxwell theory near $\mathscr{I}$. We use the standard definition of asymptotic flatness that was outlined above in section 5.1.2 (see, for instance [77]). The addition of electromagnetic fields does not spoil this definition, since $F_{ab} = \hat{F}_{ab}$ has a smooth extension to $\mathscr{I}$.

Using the conformal transformation that relates the unphysical Ricci tensor R_{ab} to the physical Ricci tensor $\hat{R}_{ab}$ (see Appendix D of [176]), the Einstein equations take the form

$$S_{ab} = -2\Omega^{-1}\nabla_a n_b + \Omega^{-2}n^c n_c g_{ab} + 8\pi\Omega^2\left(T_{ab} - \frac{1}{3}g_{ab}g^{cd}T_{cd}\right), \tag{5.86}$$

where S_{ab} and T_{ab} are given, respectively, by

$$S_{ab} \equiv R_{ab} - \frac{1}{6}Rg_{ab}, \qquad T_{ab} \equiv \Omega^{-2}\hat{T}_{ab}. \tag{5.87}$$

For electromagnetic fields, equations (5.16) and the asymptotic conditions in equations (5.30) and (5.40) imply that

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right). \tag{5.88}$$

This quantity is smooth at $\mathscr{I}$ by the smoothness of F_{ab} and g_{ab}.

Although one would expect S_{ab} to capture the radiative degrees of freedom at null infinity, this is not entirely the case. The reason for this discrepancy is two-fold: first, S_{ab} is not conformally homogeneous, and second, it does not vanish in the absence of gravitational waves. Instead, the radiative information is captured in the *news tensor* N_{ab}, which is defined by

$$N_{ab} \equiv \underleftarrow{S}_{ab} - \rho_{ab}, \tag{5.89}$$

where $\underleftarrow{S}_{ab}$ is the pullback to $\mathscr{I}$ of S_{ab}, and ρ_{ab} is the unique symmetric tensor field on $\mathscr{I}$ constructed from the universal structure at $\mathscr{I}$ in theorem 5 of [77]. The news tensor also satisfies the properties

$$N_{ab}n^b \mathrel{\widehat{=}} 0, \qquad N_{ab}q^{ab} \mathrel{\widehat{=}} 0, \tag{5.90}$$

and, as we discuss in section 5.2.3, vanishes in stationary spacetimes.

We now discuss the behavior of metric perturbations at $\mathscr{I}$. As before, we assume that the conformal factor is chosen to satisfy equations (5.31) and (5.32). Furthermore, as we used in the section on asymptotic symmetries above, without loss of generality, the conformal factor Ω in a neighborhood of $\mathscr{I}$ and the unphysical metric $g_{ab}|_{\mathscr{I}}$ at $\mathscr{I}$ may be assumed to be universal. Now, consider a physical metric perturbation $\delta\hat{g}_{ab}$. Since the conformal factor can be chosen universally, we have that

$$\delta g_{ab} = \Omega^2 \delta\hat{g}_{ab}. \tag{5.91}$$

Given that the unphysical metric $g_{ab}|_{\mathscr{I}}$ at $\mathscr{I}$ is also universal, $\delta g_{ab} \mathrel{\widehat{=}} 0$, and thus there exists a smooth tensor field τ_{ab} such that

$$\delta g_{ab} = \Omega \tau_{ab}. \tag{5.92}$$

Furthermore, imposing the Bondi condition on the perturbations, that is, $\delta(\nabla_a n_b) \mathrel{\widehat{=}} 0$, we also find [see equations (51–53) of [178]]

$$\tau_{ab}n^b = \Omega\tau_a, \tag{5.93}$$

for some smooth τ_a. Thus, our asymptotic conditions on the metric perturbations imply that the quantities

$$\tau_{ab} \equiv \Omega^{-1}\delta g_{ab}, \qquad \tau_a \equiv \Omega^{-1}\tau_{ab}n^b \tag{5.94}$$

are smooth on $\mathscr{I}$.

We finally consider another gravitational field at $\mathscr{I}$, the *asymptotic shear* of the cross-sections of $\mathscr{I}$ defined by

$$\sigma_{ab} \equiv \left(q_a{}^c q_b{}^d - \frac{1}{2} q_{ab} q^{cd} \right) \nabla_c l_d. \tag{5.95}$$

This object is a sort of "potential" for the news tensor, satisfying

$$N_{ab} = 2 \pounds_n \sigma_{ab}. \tag{5.96}$$

Moreover, the shear is directly related to τ_{ab}. Using the asymptotic conditions (5.94), the perturbation of the shear generated by the metric perturbation δg_{ab} (with fixed l_a, since l_a can be chosen independently of the spacetime) can be computed to be

$$\delta \sigma_{ab} \mathrel{\hat{=}} -\frac{1}{2} \left(q_a{}^c q_b{}^d - \frac{1}{2} q_{ab} q^{cd} \right) \tau_{cd}; \tag{5.97}$$

that is, $\delta\sigma_{ab}$ is given by the trace-free part of τ_{ab} on the cross-sections. Furthermore, from the analysis of Ashtekar and Streubel [18], $\delta\sigma_{ab}$ is equivalent to the perturbation in the equivalence class of derivatives $\{D_a\}$ defined on $\mathscr{I}$, which are the radiative degrees of freedom in vacuum GR.

This concludes the discussion of the asymptotic structure associated with Einstein-Maxwell theory that was *not* present when the background was fixed. For the electromagnetic field, we will use the same conditions as in the fixed-background theory: $A_a = \hat{A}_a$ is smooth at $\mathscr{I}$ and satisfies $n^a A_a \mathrel{\hat{=}} 0$ [equation (5.38)]. Moreover, the asymptotic symmetries, discussed in section 5.1.2.2, are the same in Einstein-Maxwell theory as they are in electromagnetism on a fixed background.

5.2.3 | Stationary solutions of the Einstein-Maxwell equations

In this section, we show that for any stationary solution $\hat{\boldsymbol{\Phi}} = (\hat{\boldsymbol{g}}, \hat{\boldsymbol{A}})$ of Einstein-Maxwell theory which is asymptotically flat, the radiative field $\mathcal{E}_a$ and the news tensor N_{ab} vanish at $\mathscr{I}$. To do so, we will first show that any nonzero timelike Killing vector field $\hat{t}^a$ in the physical spacetime gives a nonzero supertranslation t^a on $\mathscr{I}$.[9] Then, we show that this implies that $\mathcal{E}_a = 0$ on $\mathscr{I}$ for any solution of the Einstein-Maxwell equations which is stationary, satisfying $\pounds_{\hat{t}} \hat{F}_{ab} = 0$. Finally, using the proof by Geroch [77], this also implies that $N_{ab} = 0$.

[9] It can further be shown that the timelike Killing field is a BMS translation (see lemma 1.4 of [23] and also p. 54 of [77]), but we will not need this stronger result.

On $\mathscr{I}$, a supertranslation vector field takes the form $X^a \mathrel{\hat{=}} fn^a$ with $\pounds_n f \mathrel{\hat{=}} 0$. For our purposes we will also need the "subleading" form of this vector field away from $\mathscr{I}$; see, for instance, equation (21) of [78] and equation (93) of [178]. For completeness, we collect the proof in the following lemma.

Lemma 1. *Any vector field X^a in M such that $X^a|_{\mathscr{I}}$ is a BMS supertranslation is of the form*

$$X^a = fn^a - \Omega\nabla^a f + O(\Omega^2), \tag{5.98}$$

for some f smooth in M satisfying $\pounds_n f \mathrel{\hat{=}} 0$.

Proof. Since $X^a|_{\mathscr{I}}$ is a BMS supertranslation, we have that $X^a \mathrel{\hat{=}} fn^a$ for some f on $\mathscr{I}$ satisfying $\pounds_n f \mathrel{\hat{=}} 0$. We now extend the function f *arbitrarily* but smoothly into M, and so X^a takes the form

$$X^a = fn^a + \Omega Z^a, \tag{5.99}$$

for some smooth Z^a. Then, using equations (5.32) and (5.54), $\alpha_{(X)} \mathrel{\hat{=}} n_a Z^a$. Using the Bondi condition [equation (5.31)], equation (5.47) for such a vector field becomes

$$\nabla_{(a} f n_{b)} + n_{(a} Z_{b)} \mathrel{\hat{=}} n_c Z^c g_{ab}. \tag{5.100}$$

Taking the trace on both sides gives $n_a Z^a \mathrel{\hat{=}} 0$ (using the condition $\pounds_n f = 0$), and so we find that the right-hand side vanishes. As such, the left-hand side can only hold if $Z_a = -\nabla_a f$. □

Note that we extended the function f away from $\mathscr{I}$ in an arbitrary manner. It is easy to check from equation (5.98) that the freedom in this extension affects only the $O(\Omega^2)$ part of the vector field, which is not relevant to the discussion in this chapter.

We now turn to timelike Killing fields of the physical spacetime $(\hat{M}, \hat{g}_{ab})$, and show that they correspond to nontrivial supertranslations on null infinity.

Lemma 2. *Let $\hat{t}^a$ be a nonzero timelike Killing vector field in the physical spacetime $(\hat{M}, \hat{g}_{ab})$. Then $t^a = \hat{t}^a$ is a nonzero supertranslation on $\mathscr{I}$.*

Proof. Since $\pounds_{\hat{t}} \hat{g}_{ab} = 0$, from equation (5.44) it follows that $t^a = \hat{t}^a$ is a BMS vector field on $\mathscr{I}$. Since $\hat{t}^a$ is timelike in the physical spacetime, we have $\hat{g}_{ab}\hat{t}^a\hat{t}^b < 0$. In the unphysical spacetime away from null infinity (that is, on $M - \mathscr{I}$), this gives $\Omega^{-2} g_{ab} t^a t^b < 0$. Now, $\Omega > 0$ on $M - \mathscr{I}$, $\Omega \mathrel{\hat{=}} 0$, and g_{ab} and t^a extend smoothly to $\mathscr{I}$, and so

$$g_{ab} t^a t^b \leq 0 \tag{5.101}$$

in M, with the equality possibly holding on $\mathscr{I}$. Writing $t^a \mathrel{\hat{=}} \beta n^a + Y^a$ [from equation (5.55)], we get that $q_{ab} Y^a Y^b \leq 0$ on $\mathscr{I}$. Since q_{ab} is a Riemannian metric on the cross-sections of $\mathscr{I}$ and Y^a is tangent to these cross-sections, this means $Y^a \mathrel{\hat{=}} 0$. Thus the "Lorentz part" of t^a vanishes and t^a is a BMS supertranslation—in fact, this means that t^a is *null* on $\mathscr{I}$.

Next, we show that this supertranslation is necessarily nonzero on $\mathscr{I}$ (see also [23]). We will proceed by assuming that $t^a \mathrel{\hat{=}} 0$ and show that this implies that $\hat{t}^a$ vanishes everywhere, contradicting the assumption that it is a nonzero Killing vector field. Since t^a is a supertranslation on $\mathscr{I}$, if $t^a \mathrel{\hat{=}} 0$, then from lemma 1, we have that

$$t^a = \Omega^2 W^a, \tag{5.102}$$

for some smooth W^a. Since $\hat{t}^a$ is a Killing vector field in the physical spacetime $(\hat{M}, \hat{g}_{ab})$, t^a is a conformal Killing field in the unphysical spacetime (M, g_{ab}), with

$$\pounds_t g_{ab} = 2\alpha_{(t)} g_{ab}, \qquad \alpha_{(t)} = \Omega^{-1} n_a t^a. \tag{5.103}$$

Any conformal Killing field is completely determined by its *conformal Killing data* X^A, specified at any point $p \in M$ (see, for example, [21]):

$$X^A \equiv \begin{pmatrix} X^a \\ \nabla_{[a} X_{b]} \\ \alpha_{(X)} \\ \nabla_a \alpha_{(X)} \end{pmatrix} \tag{5.104}$$

(here, we are using a similar notation as in section 4.1.3.2 of the previous chapter: this vector can be considered as a section of a "conformal Killing vector bundle"). Furthermore, if X^A vanishes at any point p, then the corresponding conformal Killing vector field X^a vanishes *everywhere*. We now show that t^A, using the conformal Killing vector field t^a in (5.102), vanishes on $\mathscr{I}$. It is easy to see by a direct computation that t^a, $\nabla_{[a}t_{b]}$, and $\alpha_{(t)}$ vanish on $\mathscr{I}$. Computing the last remaining piece of t^A, we have that

$$\nabla_a \alpha_{(t)} \mathrel{\hat{=}} n_a(n_b W^b). \tag{5.105}$$

To show that this vanishes at $\mathscr{I}$, we evaluate $\pounds_t g_{ab} = 2\alpha_{(t)} g_{ab}$ with (5.102) to obtain

$$4\Omega n_{(a}W_{b)} + 2\Omega^2 \nabla_{(a}W_{b)} = 2\Omega n_c W^c g_{ab}. \tag{5.106}$$

Note that this holds in a neighborhood of $\mathscr{I}$, not just on $\mathscr{I}$, as a consequence of $\hat{t}^a$ being Killing in the physical spacetime. Multiplying the above equation by Ω^{-1}, taking the trace, and then taking the limit to $\mathscr{I}$, we get $n_a W^a \mathrel{\hat{=}} 0$, and so $\nabla_a \alpha_{(t)} \mathrel{\hat{=}} 0$. Hence, t^A vanishes on $\mathscr{I}$, and thus $t^a = 0$ everywhere in M. This implies that $\hat{t}^a = 0$ in $\hat{M}$, which contradicts the assumption that $\hat{t}^a$ is a nonzero Killing field in the physical spacetime. Thus, any nonzero timelike Killing vector field in the physical spacetime is necessarily a nonzero supertranslation on $\mathscr{I}$. □

Finally, we show that, for a stationary solution of Einstein-Maxwell theory, the radiative fields N_{ab} and $\mathcal{E}_a$ vanish on null infinity.[10]

Theorem 1. *Let $\hat{\boldsymbol{\Phi}} = (\hat{\boldsymbol{g}}, \hat{\boldsymbol{A}})$ be a stationary solution of Einstein-Maxwell theory, that is, there exists a timelike vector field $\hat{t}^a$ in the physical spacetime $\hat{M}$ such that*

$$\pounds_{\hat{t}} \hat{g}_{ab} = 0, \qquad \pounds_{\hat{t}} \hat{F}_{ab} = 0. \tag{5.107}$$

It then follows that the radiative fields vanish on $\mathscr{I}$: $N_{ab} \mathrel{\hat{=}} 0$ and $\mathcal{E}_a \mathrel{\hat{=}} 0$.

Proof. Consider first the stationary electromagnetic field $\hat{F}_{ab}$, which satisfies in the unphysical spacetime $\pounds_t F_{ab} = 0$, where $t^a = \hat{t}^a$, as above. From lemmas 1 and 2, we have that

$$t^a = f n^a - \Omega \nabla^a f + O(\Omega^2), \tag{5.108}$$

for some $f \neq 0$ satisfying $\pounds_n f \mathrel{\hat{=}} 0$. Evaluating the pullback of $\pounds_t F_{ab} n^b = 0$ to $\mathscr{I}$ and using $\pounds_t n^a \mathrel{\hat{=}} 0$ and $\pounds_n f \mathrel{\hat{=}} 0$ (as t^a is a supertranslation) gives

$$\pounds_n (f \mathcal{E}_a) \mathrel{\hat{=}} 0. \tag{5.109}$$

[10] Note that for this result to hold it is essential that the space of generators of $\mathscr{I}$ is topologically $\mathbb{S}^2$.

As such, $f\mathcal{E}_a$ can be considered to be a covector on the space of generators of $\mathscr{I}$. Similarly, evaluating the pullback of $\pounds_t F_{ab} = 0$ to $\mathscr{I}$, we have

$$\mathscr{D}_{[a}(f\mathcal{E}_{b]}) \mathrel{\hat{=}} 0. \tag{5.110}$$

Note that only the derivative along the cross-sections $\mathscr{D}_a$ occurs in this equation due to equation (5.109) and the Bondi condition [equation (5.31)]. Next, evaluating $l^a n^b \pounds_t F_{ab} \mathrel{\hat{=}} 0$, we have

$$\begin{aligned} 0 \mathrel{\hat{=}} l^a n^b \pounds_t F_{ab} &\mathrel{\hat{=}} \pounds_t (F_{ab} l^a n^b) - F_{ab} \pounds_t l^a n^b \\ &\mathrel{\hat{=}} f \pounds_n (F_{ab} l^a n^b) + F_{ab}(n^a \pounds_l f + \nabla^a f) n^b \\ &\mathrel{\hat{=}} f q^{ab} \mathscr{D}_a \mathcal{E}_b + q^{ab} \mathcal{E}_a \mathscr{D}_b f \\ &\mathrel{\hat{=}} q^{ab} \mathscr{D}_a (f \mathcal{E}_b), \end{aligned} \tag{5.111}$$

where the first line uses $\pounds_t n^a \mathrel{\hat{=}} 0$ for a supertranslation, the second line is a straightforward computation using equation (5.108), and the third line uses the Maxwell equation (5.43). Equations (5.110) and (5.111) imply that $f\mathcal{E}_a$, as a covector field on the space of generators of $\mathscr{I}$, has vanishing curl and divergence. Since the space of generators of $\mathscr{I}$ is topologically $\mathbb{S}^2$ and $f \neq 0$, this implies that $\mathcal{E}_a = 0$ for any stationary solution.

Now, equation (5.88) implies that $T_{ab} n^a n^b \mathrel{\hat{=}} \frac{1}{4\pi} \mathcal{E}_a \mathcal{E}^a$, and thus for any stationary solution, $T_{ab} n^a n^b \mathrel{\hat{=}} 0$ as well. With this condition and the Einstein equation, it can be shown that $N_{ab} \mathrel{\hat{=}} 0$ for any stationary spacetime (see pp. 53–54 of [77]). Thus, for any stationary solution of the Einstein-Maxwell equations, we have $N_{ab} \mathrel{\hat{=}} 0$ and $\mathcal{E}_a \mathrel{\hat{=}} 0$, as desired. □

5.3 The Wald-Zoupas Prescription

In this section, we derive the charges and fluxes associated with asymptotic symmetries in Einstein-Maxwell theory at null infinity using the Wald-Zoupas prescription. In section 5.3.1, we first review the Wald-Zoupas procedure for obtaining charges and fluxes corresponding to asymptotic symmetries for a general diffeomorphism-covariant theory. We then apply this prescription to the Einstein-Maxwell case in section 5.3.2. Of the three currents for electromagnetic fields on a fixed background, we show that the contribution of the electromagnetic fields to the Wald-Zoupas flux is given not by stress-energy current, but by the Noether current. Furthermore, as expected from the Wald-Zoupas procedure, this flux can be determined entirely from the radiative degrees of freedom, and the total flux over all of $\mathscr{I}$ acts as a Hamiltonian generator on the radiative phase space.

5.3.1 Formalism

The prescription of Wald and Zoupas [178] provides a method of determining charges and fluxes at null infinity, and can be applied to any local and covariant theory. We review below the essential ingredients, emphasizing the subsequent application to Einstein-Maxwell theory.

When the dynamical fields $\hat{\mathbf{\Phi}}$ satisfy the equations of motion, and $\delta\hat{\mathbf{\Phi}}$ satisfy the linearized equations of motion, one can show that (see [113, 104, 133])

$$\boldsymbol{\omega}[\hat{\mathbf{\Phi}};\delta\hat{\mathbf{\Phi}},\delta_{\hat{\xi}}\hat{\mathbf{\Phi}}] = \mathrm{d}\left\{\delta\boldsymbol{Q}[\hat{\boldsymbol{\xi}}] - \hat{\boldsymbol{X}}\cdot\boldsymbol{\theta}[\hat{\mathbf{\Phi}};\boldsymbol{\delta}\hat{\mathbf{\Phi}}]\right\}, \tag{5.112}$$

for all symmetries $\hat{\boldsymbol{\xi}}$, where the 2-form $\boldsymbol{Q}[\hat{\boldsymbol{\xi}}]$ is the *Noether charge* associated with the symmetry $\hat{\boldsymbol{\xi}}$. In Einstein-Maxwell theory, $\boldsymbol{Q}[\hat{\boldsymbol{\xi}}]$ is given by

$$\boldsymbol{Q}[\hat{\boldsymbol{\xi}}] \equiv -\frac{1}{8\pi}{}^{\star}\mathrm{d}\hat{\boldsymbol{X}} - \frac{1}{4\pi}{}^{\star}\hat{\boldsymbol{F}}(\hat{\boldsymbol{X}}\cdot\hat{\boldsymbol{A}} + \hat{\lambda}). \tag{5.113}$$

The first term above is the Noether charge associated with the vector field $\hat{X}^a$ in vacuum general relativity [equation (44) of [178]], and the second term is the Noether charge for electromagnetism given in equation (5.25).

Now we consider equation (5.112) at $\mathscr{I}$, rewritten in terms of the unphysical fields which are smooth at $\mathscr{I}$. Using equations (5.94), (5.30), and (5.40), it can be verified that the symplectic current in equation (5.81) has a limit to $\mathscr{I}$. Thus, from this point onward, we work with the fields and symmetries in the unphysical spacetime. Now, consider a spacelike surface Σ which intersects $\mathscr{I}$ at some cross-section S. Integrating equation (5.112) over Σ, we then find

$$\int_\Sigma \boldsymbol{\omega}[\mathbf{\Phi};\delta\mathbf{\Phi},\delta_\xi\mathbf{\Phi}] = \int_S \left\{\delta\boldsymbol{Q}[\boldsymbol{\xi}] - \boldsymbol{X}\cdot\boldsymbol{\theta}[\mathbf{\Phi};\delta\mathbf{\Phi}]\right\}. \tag{5.114}$$

Since the symplectic current admits a limit to $\mathscr{I}$, the integral on the left-hand side of equation (5.114) is always finite. However, the 2-form integrand on the right-hand side need not have a finite limit to $\mathscr{I}$ in general. Thus, the integral on the right-hand side of equation (5.114) should be understood as being defined by first integrating over some 2-sphere in Σ and then taking the limit of this 2-sphere to S [178]. This final limiting integral is independent of the way in which the limits are taken since $\mathrm{d}\boldsymbol{\omega}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi},\delta_\xi\boldsymbol{\Phi}]=0$.

From the above identity, it would be natural to define a charge associated with the asymptotic symmetry ξ at S as a function $Q[\boldsymbol{\xi};S]$ in the phase space of the theory such that

$$\delta Q[\boldsymbol{\xi};S]=\int_S\left\{\delta\boldsymbol{Q}[\boldsymbol{\xi}]-\boldsymbol{X}\cdot\boldsymbol{\theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]\right\}. \tag{5.115}$$

for all perturbations $\delta\Phi$. However, in general, no such charge exists, since the right-hand side is not integrable in phase space; that is, it cannot be written as the variation of some quantity for *all* perturbations. To see this, suppose that the charge defined in equation (5.115) does exist. Then, one must have $(\delta_1\delta_2-\delta_2\delta_1)Q[\boldsymbol{\xi};S]=0$ for *all* backgrounds $\boldsymbol{\Phi}$ and *all* perturbations $\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}$ (satisfying the corresponding equations of motion). However, it is straightforward to compute that

$$(\delta_1\delta_2-\delta_2\delta_1)Q[\boldsymbol{\xi};S]=-\int_S\boldsymbol{X}\cdot\boldsymbol{\omega}[\boldsymbol{\Phi};\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}]. \tag{5.116}$$

Thus, a charge defined by equation (5.115) will exist if the right-hand side of the above equation vanishes. This is the case in Einstein-Maxwell theory if $X^a\,\widehat{=}\,0$ (that is, for a pure asymptotic gauge symmetry), or if X^a is tangent to S. However, in general, the right-hand side is non-vanishing, and so one cannot define any charge $Q[\boldsymbol{\xi};S]$ using equation (5.115).

This obstruction is resolved by the rather general prescription of Wald and Zoupas [178]. Their procedure for defining integrable charges associated with asymptotic symmetries can be summarized as follows: let $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$ be a symplectic potential for the pullback of the symplectic current to $\mathscr{I}$; that is,

$$\underleftarrow{\boldsymbol{\omega}}[\boldsymbol{\Phi};\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}]=\delta_1\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta_2\boldsymbol{\Phi}]-\delta_2\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta_1\boldsymbol{\Phi}], \tag{5.117}$$

for *all* backgrounds and *all* perturbations, with suitable asymptotic conditions and equations of motion imposed. Following [178], we require the following properties of $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$:

1. $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$ must be locally and covariantly constructed out of the dynamical fields $\boldsymbol{\Phi}$, $\delta\boldsymbol{\Phi}$, and finitely many of their derivatives, along with any fields in the "universal background structure" present at $\mathscr{I}$;

2. $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$ must be independent of any arbitrary choices made in specifying the background structure; that is, $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$ is conformally invariant as well as invariant under gauge trans-

formations on $\mathscr{I}$ for Einstein-Maxwell theory; we also require that $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ be independent of the choice of the auxiliary normal l^a and the corresponding q^{ab} used in our computations; and

3. if $\mathbf{\Phi}$ is a stationary background solution, then $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}] = 0$, for *all* (not necessarily stationary) perturbations $\delta\mathbf{\Phi}$.

If such a symplectic potential $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ can be found, define $\mathcal{Q}[\boldsymbol{\xi}; S]$ to be a function on the phase space at $\mathscr{I}$ by[11]

$$\delta\mathcal{Q}[\boldsymbol{\xi}; S] \equiv \int_S \{\delta\boldsymbol{Q}[\boldsymbol{\xi}] - \boldsymbol{X}\cdot\boldsymbol{\theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]\} + \int_S \boldsymbol{X}\cdot\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]. \tag{5.118}$$

It can easily be checked [using equations (5.115), (5.116), and (5.117)] that this expression is integrable in phase space; that is, $(\delta_1\delta_2 - \delta_2\delta_1)\mathcal{Q}[\boldsymbol{\xi}; S] = 0$. Together with some choice of reference solution $\mathbf{\Phi}_0$ on which $\mathcal{Q}[\boldsymbol{\xi}; S] = 0$ for all asymptotic symmetries $\boldsymbol{\xi}$ and all cross-sections S, equation (5.118) can be integrated in phase space to define the *Wald-Zoupas charge* $\mathcal{Q}[\boldsymbol{\xi}; S]$ associated with the asymptotic symmetry $\boldsymbol{\xi}$ at S.

The flux of the perturbed Wald-Zoupas charge is given by [see equations (28) and (29) of [178]]

$$\delta\mathcal{F}[\boldsymbol{\xi}; \Delta\mathscr{I}] \equiv \delta\mathcal{Q}[\boldsymbol{\xi}; S_2] - \delta\mathcal{Q}[\boldsymbol{\xi}; S_1] = -\int_{\Delta\mathscr{I}} \Big(\underleftarrow{\boldsymbol{\omega}}[\mathbf{\Phi}; \delta\mathbf{\Phi}, \delta_\xi\mathbf{\Phi}] + \mathrm{d}\{\boldsymbol{X}\cdot\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]\}\Big). \tag{5.119}$$

The last term of this equation can also be written as

$$\begin{aligned}\mathrm{d}\{\boldsymbol{X}\cdot\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]\} &= \pounds_X \mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}] \\ &= \delta_\xi\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}] \\ &= -\underleftarrow{\boldsymbol{\omega}}[\mathbf{\Phi}; \delta\mathbf{\Phi}, \delta_\xi\mathbf{\Phi}] + \delta\mathbf{\Theta}[\mathbf{\Phi}; \delta_\xi\mathbf{\Phi}],\end{aligned} \tag{5.120}$$

where in the second line we have used the criteria that $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ is a local and covariant functional on $\mathscr{I}$ and that it is invariant under gauge transformations,[12] while the third line follows from the definition of $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ as a symplectic potential for $\underleftarrow{\boldsymbol{\omega}}[\mathbf{\Phi}; \delta_1\mathbf{\Phi}, \delta_2\mathbf{\Phi}]$ [equation (5.117)]. The flux of the perturbed Wald-Zoupas charge is therefore simply given by

$$\delta\mathcal{F}[\boldsymbol{\xi}; \Delta\mathscr{I}] = -\int_{\Delta\mathscr{I}} \delta\mathbf{\Theta}[\mathbf{\Phi}; \delta_\xi\mathbf{\Phi}]. \tag{5.121}$$

[11] Note that the first of these two integrals is defined by the limiting procedure described below equation (5.114), whereas the second is an ordinary integral, as $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ is defined directly on $\mathscr{I}$.

[12] In the principal bundle language, this means $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ is a gauge-invariant and *horizontal* 3-form on the bundle.

To get the unperturbed charge and flux, we have to choose a reference solution $\mathbf{\Phi}_0$ on which the charges are required to vanish. Since the $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ is required to vanish on stationary backgrounds, we choose the reference solution $\mathbf{\Phi}_0$ to also be stationary. For our concrete case of Einstein-Maxwell theory, we will pick $\mathbf{\Phi}_0$ to be Minkowski spacetime. Then, the flux of the Wald-Zoupas charge is simply

$$\mathcal{F}[\boldsymbol{\xi}; \Delta\mathscr{I}] = \mathcal{Q}[\boldsymbol{\xi}; S_2] - \mathcal{Q}[\boldsymbol{\xi}; S_1] = -\int_{\Delta\mathscr{I}} \mathbf{\Theta}[\mathbf{\Phi}; \delta_\xi \mathbf{\Phi}]. \tag{5.122}$$

Note that $\mathbf{\Theta}[\mathbf{\Phi}; \delta_\xi \mathbf{\Phi}]$ can only depend upon radiative degrees of freedom, for the following reason: the pullback of the symplectic product *defines* the radiative phase space of the theory, and so its symplectic potential (if it vanishes on non-radiative solutions) can only depend on radiative degrees of freedom. As such, the flux of the Wald-Zoupas charge is always purely radiative.

Note that, from equation (5.119), we also have

$$\delta\mathcal{F}[\boldsymbol{\xi}; \Delta\mathscr{I}] = -\int_{\Delta\mathscr{I}} \underleftarrow{\boldsymbol{\omega}}[\mathbf{\Phi}; \delta\mathbf{\Phi}, \delta_\xi\mathbf{\Phi}] + \int_{S_2} \boldsymbol{X} \cdot \mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}] - \int_{S_1} \boldsymbol{X} \cdot \mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]. \tag{5.123}$$

If the boundary terms on S_2 and S_1 vanish for all backgrounds $\mathbf{\Phi}$ and all perturbations $\delta\mathbf{\Phi}$, then $\mathcal{F}[\boldsymbol{\xi}; \Delta\mathscr{I}]$ also defines a *Hamiltonian generator* (relative to the symplectic current $\underleftarrow{\boldsymbol{\omega}}[\mathbf{\Phi}; \delta_1\mathbf{\Phi}, \delta_2\mathbf{\Phi}]$) on the radiative phase space on $\Delta\mathscr{I}$ corresponding to the symmetry $\boldsymbol{\xi}$. For general field configurations, these boundary terms do not vanish on finite cross-sections of $\mathscr{I}$. However, we will show below in Einstein-Maxwell theory that when $\Delta\mathscr{I}$ is taken to be all of null infinity, appropriate boundary conditions at timelike and spacelike infinity (that is, as $|u| \to \infty$) ensure that these boundary terms indeed vanish for our choice of $\mathbf{\Theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$. Thus, our fluxes define the Hamiltonian generators for Einstein-Maxwell theory on the phase space on all of $\mathscr{I}$.

We conclude this section with a discussion of ambiguities in the Wald-Zoupas prescription. For a given Lagrangian $\boldsymbol{L}$, the symplectic potential $\boldsymbol{\theta}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ is ambiguous up to the redefinition

$$\boldsymbol{\theta}[\hat{\mathbf{\Phi}}; \delta\hat{\mathbf{\Phi}}] \mapsto \boldsymbol{\theta}[\hat{\mathbf{\Phi}}; \delta\hat{\mathbf{\Phi}}] + \mathrm{d}\boldsymbol{Y}[\hat{\mathbf{\Phi}}; \delta\hat{\mathbf{\Phi}}], \tag{5.124}$$

where $\boldsymbol{Y}[\mathbf{\Phi}; \delta\mathbf{\Phi}]$ is a local and covariant 2-form which is a linear functional of the perturbation $\delta\hat{\mathbf{\Phi}}$ and finitely many of its derivatives. This changes the symplectic current by

$$\boldsymbol{\omega}[\hat{\mathbf{\Phi}}; \delta_1\hat{\mathbf{\Phi}}, \delta_2\hat{\mathbf{\Phi}}] \mapsto \boldsymbol{\omega}[\hat{\mathbf{\Phi}}; \delta_1\hat{\mathbf{\Phi}}, \delta_2\hat{\mathbf{\Phi}}] + \mathrm{d}\left\{\delta_1\boldsymbol{Y}[\hat{\mathbf{\Phi}}; \delta_2\hat{\mathbf{\Phi}}] - \delta_2\boldsymbol{Y}[\delta_1\hat{\mathbf{\Phi}}]\right\}. \tag{5.125}$$

Note that the addition of a boundary term to the Lagrangian does not affect the symplectic form. Even with a fixed choice of the symplectic current, the symplectic potential $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$ defined on null infinity by equation (5.117) is ambiguous up to

$$\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}] \mapsto \boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}] + \delta\boldsymbol{W}[\boldsymbol{\Phi}], \tag{5.126}$$

where $\boldsymbol{W}[\boldsymbol{\Phi}]$ is a local and covariant 3-form on $\mathscr{I}$. These ambiguities then also lead to ambiguities in the Wald-Zoupas prescription for the charges and fluxes on null infinity. It was argued by Wald and Zoupas that these ambiguities do not affect their prescription in vacuum GR [see footnote 18 and the arguments below equation (73) in [178]]. We hope that similar arguments can also be made for Einstein-Maxwell theory, but we do not analyze these ambiguities in detail.

5.3.2 Einstein-Maxwell theory

In this section, we apply the above described prescription of Wald and Zoupas to Einstein-Maxwell theory and compute the charges and fluxes at $\mathscr{I}$. In particular, we find that the charges at $\mathscr{I}$ contain a contribution solely from the electromagnetic fields, and we then compute these contributions in a few select examples.

5.3.2.1 Charges and Fluxes

We first compute the full Wald-Zoupas charges and fluxes for Einstein-Maxwell theory. Since our main focus is on the contribution of the electromagnetic fields to the charges and fluxes, we will borrow the analysis of Wald and Zoupas [178] for the contribution of the gravitational field.

We start with the pullback to $\mathscr{I}$ of the symplectic current in equation (5.81). Using the asymptotic conditions in equations (5.94), (5.30), and (5.40), it can be checked that the contribution from the cross-term given by $-\Omega^{-4} n_a \hat{w}^a_\times[\boldsymbol{\Phi};\delta_1\boldsymbol{\Phi};\delta_2\boldsymbol{\Phi}]$ [equation (5.85)] vanishes in the limit to $\mathscr{I}$. The contribution from the electromagnetic fields is easily computed to be

$$\underleftarrow{\boldsymbol{\omega}_{\rm EM}}[\delta\boldsymbol{g};\delta_1\boldsymbol{A},\delta_2\boldsymbol{A}] \mathrel{\hat{=}} -\Omega^{-4} n_a \hat{w}^a_{\rm EM}[\delta\boldsymbol{g};\delta_1\boldsymbol{A},\delta_2\boldsymbol{A}]\boldsymbol{\epsilon}_3 = -\frac{1}{4\pi}\left[\delta_1\mathcal{E}^a\delta_2 A_a - (1\leftrightarrow 2)\right]\boldsymbol{\epsilon}_3. \tag{5.127}$$

The contribution from the metric perturbations is the most tedious to compute. However, since T_{ab} is smooth on $\mathscr{I}$, the terms proportional to the stress-energy tensor in (5.86) vanish at $\mathscr{I}$, and the computation of [178] carries over unchanged. We therefore find [see equation (72) of [178]][13]

$$\underset{\leftarrow}{\boldsymbol{\omega}}_{\text{GR}}[\boldsymbol{g};\delta_1\boldsymbol{g},\delta_2\boldsymbol{g}] \hat{=} -\Omega^{-4}n_a\hat{w}^a_{\text{GR}}[\boldsymbol{g};\delta_1\boldsymbol{g},\delta_2\boldsymbol{g}]\boldsymbol{\epsilon}_3 = -\frac{1}{32\pi}\left[\delta_1 N_{ab}\tau_2^{ab} - (1\leftrightarrow 2)\right]\boldsymbol{\epsilon}_3. \tag{5.128}$$

Thus, the pullback to $\mathscr{I}$ of the symplectic current of Einstein-Maxwell theory is given by

$$\underset{\leftarrow}{\boldsymbol{\omega}}[\boldsymbol{\Phi};\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}] = -\frac{1}{32\pi}\left[\delta_1 N_{ab}\tau_2^{ab} - (1\leftrightarrow 2)\right]\boldsymbol{\epsilon}_3 - \frac{1}{4\pi}\left[\delta_1\mathcal{E}^a\delta_2 A_a - (1\leftrightarrow 2)\right]\boldsymbol{\epsilon}_3. \tag{5.129}$$

Note that $\underset{\leftarrow}{\boldsymbol{\omega}}[\boldsymbol{\Phi};\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}]$ is determined completely by the (perturbed) radiative degrees of freedom. For the electromagnetic fields, it is clear that only the perturbations of $\underset{\leftarrow}{A}_a$ and $\mathcal{E}_a = -\pounds_n\underset{\leftarrow}{A}_a$ contribute. For the gravitational fields, the argument is more involved, but due to the conditions (5.90) and $\tau_{ab}n^b \hat{=} 0$ [from equation (5.94)], it is clear that only this trace-free part of τ_{ab}—equivalently, $\delta\sigma_{ab}$—contributes to the pullback of the symplectic current. This quantity, as discussed above in section 5.2.2, encodes the radiative degrees of freedom in vacuum GR. Thus, $\underset{\leftarrow}{\boldsymbol{\omega}}[\boldsymbol{\Phi};\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}]$ is completely determined by the perturbed radiative degrees of freedom in Einstein-Maxwell theory. The integral of this symplectic current over all of $\mathscr{I}$ [when appropriate falloff conditions are satisfied toward i^0 and i^+; see equation (5.140)] reproduces the symplectic form on the radiative phase space at null infinity used by Ashtekar and Streubel [18].

To apply the Wald-Zoupas prescription, we need to find a 3-form symplectic potential $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$ for $\underset{\leftarrow}{\boldsymbol{\omega}}[\boldsymbol{\Phi};\delta_1\boldsymbol{\Phi},\delta_2\boldsymbol{\Phi}]$ given in equation (5.129). We choose the following (note the ambiguities in the choice of $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$ that we gave at the end of section 5.3.1):

$$\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}] = \boldsymbol{\Theta}_{\text{GR}}[\boldsymbol{g};\delta\boldsymbol{g}] + \boldsymbol{\Theta}_{\text{EM}}[\boldsymbol{A};\delta\boldsymbol{A}] \tag{5.130}$$

where

$$\boldsymbol{\Theta}_{\text{GR}}[\boldsymbol{g};\delta\boldsymbol{g}] = -\frac{1}{32\pi}N_{ab}\tau^{ab}\boldsymbol{\epsilon}_3, \qquad \boldsymbol{\Theta}_{\text{EM}}[\boldsymbol{A};\delta\boldsymbol{A}] = -\frac{1}{4\pi}\mathcal{E}^a\delta A_a\boldsymbol{\epsilon}_3. \tag{5.131}$$

Note that $\boldsymbol{\Theta}_{\text{GR}}[\boldsymbol{g};\delta\boldsymbol{g}]$ is the symplectic potential for vacuum GR given in equation (73) of [178]. The above choice of $\boldsymbol{\Theta}$ satisfies all the requirements listed below (5.117):

1. the $\boldsymbol{\Theta}[\boldsymbol{\Phi};\delta\boldsymbol{\Phi}]$ in (5.131) is indeed a local and covariant functional of the background fields $\boldsymbol{\Phi}$ and the perturbed fields $\delta\boldsymbol{\Phi}$ (see also footnote 20 of [178] for an explanation of the locality of the news tensor);

[13] As mentioned before, one can consider additional sources with compact support or sufficient falloff at $\mathscr{I}$ without affecting this analysis.

2. it is also invariant under conformal transformations and gauge transformations,[14] and the choice of the auxiliary null normal l^a and the "inverse metric" q^{ab};

3. as we showed in 5.2.3, for stationary solutions of Einstein-Maxwell theory we have $\mathcal{E}_a = 0$ and $N_{ab} = 0$ on $\mathscr{I}$, and thus $\boldsymbol{\Theta}[\boldsymbol{\Phi}; \delta\boldsymbol{\Phi}]$, as defined above, vanishes for all perturbations $\delta\boldsymbol{\Phi}$ whenever the background $\boldsymbol{\Phi}$ is a stationary solution of the Einstein-Maxwell equations.

Having chosen a $\boldsymbol{\Theta}[\boldsymbol{\Phi}; \delta\boldsymbol{\Phi}]$ as in equation (5.131), the Wald-Zoupas flux $\mathcal{F}[\boldsymbol{\xi}; \Delta\mathscr{I}]$ that is associated with an asymptotic symmetry $\boldsymbol{\xi}$ is determined by equation (5.122). We now want to find the corresponding Wald-Zoupas charge $\mathcal{Q}[\boldsymbol{\xi}; S]$ on any cross-section S of $\mathscr{I}$. Note that the Wald-Zoupas charge is determined by (5.118), along with the requirement that it vanish on some stationary reference solution $\boldsymbol{\Phi}_0$, which we take to be Minkowski spacetime. Although the right-hand side of (5.118) can be directly computed, it is not very useful to find an expression for $\mathcal{Q}[\boldsymbol{\xi}; S]$. We instead proceed in the following manner: let the Wald-Zoupas charge be given by

$$\mathcal{Q}[\boldsymbol{\xi}; S] = \mathcal{Q}_{\mathrm{GR}}[\boldsymbol{X}; S] + \mathcal{Q}_{\mathrm{EM}}[\boldsymbol{\xi}; S], \tag{5.132}$$

where $\mathcal{Q}_{\mathrm{GR}}[\boldsymbol{X}; S]$ is the expression for the charge in vacuum GR [see equation (5.133)], and $\mathcal{Q}_{\mathrm{EM}}[\boldsymbol{\xi}; S]$ is the (as yet undetermined) contribution due to the electromagnetic fields. As we will show below, in the presence of electromagnetic fields, $\mathcal{Q}_{\mathrm{GR}}[\boldsymbol{X}; S]$ by itself does not satisfy (5.122) with $\boldsymbol{\Theta}[\boldsymbol{\Phi}; \delta\boldsymbol{\Phi}]$ as in (5.131); that is, $\mathcal{Q}_{\mathrm{GR}}[\boldsymbol{X}; S]$ is not the full Wald-Zoupas charge for Einstein-Maxwell theory. Then, we will define the electromagnetic contribution $\mathcal{Q}_{\mathrm{EM}}[\boldsymbol{\xi}; S]$ so that the total charge (5.132) *does* satisfy equations (5.122) and (5.131), and $\mathcal{Q}_{\mathrm{EM}}[\boldsymbol{\xi}; S]$ vanishes in the absence of the electromagnetic field.

In vacuum GR, the Wald-Zoupas charge for a BMS vector field X^a can be written as follows. With our assumptions on the asymptotic conditions on the fields, it follows that $C_{abcd} \mathrel{\widehat{=}} 0$ (see theorem 11 of [77]), and thus $\Omega^{-1} C_{abcd}$ is smooth at $\mathscr{I}$. The charge $\mathcal{Q}_{\mathrm{GR}}[\boldsymbol{X}; S]$ is given by

$$\mathcal{Q}_{\mathrm{GR}}[\boldsymbol{X}; S] = \frac{1}{8\pi} \int_S \boldsymbol{\epsilon}_2 \left[-X^a (\Omega^{-1} C_{abcd}) l^b l^c n^d + \frac{1}{2} \beta \sigma^{ab} N_{ab} + Y^a \sigma_{ab} \mathscr{D}_c \sigma^{bc} - \frac{1}{4} \sigma_{ab} \sigma^{ab} \mathscr{D}_c Y^c \right], \tag{5.133}$$

[14] Note that δA_a is gauge invariant, since A_a (the background field) and $A_a(\lambda)$ *both* shift by Λ under a gauge transformation.

where we have decomposed $X^a \mathrel{\hat{=}} \beta n^a + Y^a$, with Y^a tangent to the cross-sections of the chosen foliation, as in equation (5.55). The tensor σ_{ab} is the asymptotic shear of the cross-sections defined in (5.95).

For *vacuum* GR, the charge expression (5.133) coincides with the charges defined by Wald and Zoupas [178]. Showing this explicitly is a long and tedious computation, but we argue as follows. For supertranslations, (5.133) is the same as the supermomentum defined by Geroch [77], which is equal to the Wald-Zoupas charge [see equation (98) of [178]]. For asymptotic Lorentz symmetries, it was shown in [178] that the Wald-Zoupas charge is given by the "linkage" charge[15] found by Geroch and Winicour [78], which, in turn, coincides with the above expression as shown by Winicour [184]. The expression (5.133) is also equal to the charge found in [109], when the conformal factor is additionally chosen away from $\mathscr{I}$ to make the vector field l^a expansion-free. It is also equal to the expression computed using Bondi coordinates [see, for instance, equation (35) of [72]].

In vacuum GR, the flux of the charge (5.133) is given by equation (5.122), with $\boldsymbol{\Theta}_{\text{GR}}[\boldsymbol{g}; \pounds_X \boldsymbol{g}]$ on the right-hand side. However, in the presence of electromagnetic fields, one gets an additional contribution to the flux of this charge through the asymptotic stress-energy tensor T_{ab}. This additional contribution arises through the $\pounds_n$ of the Weyl tensor term, and using the Bianchi identity at $\mathscr{I}$ we get[16]

$$\mathcal{Q}_{\text{GR}}[\boldsymbol{X}; S_2] - \mathcal{Q}_{\text{GR}}[\boldsymbol{X}; S_1] = -\int_{\Delta\mathscr{I}} \Big[\boldsymbol{\Theta}_{\text{GR}}[\boldsymbol{g}; \pounds_X \boldsymbol{g}] + T_{ab} n^a X^b \boldsymbol{\epsilon}_3\Big] . \tag{5.134}$$

If one takes $\mathcal{Q}_{\text{GR}}[\boldsymbol{X}; S]$ as the definition of the charges associated with the BMS symmetries, then the electromagnetic fields contribute to the flux only through the asymptotic stress-energy tensor T_{ab} (see also Appendix C of [72]). As argued in section 5.1.3 and in [20, 19], for Lorentz symmetries this contribution to the flux is *not* purely radiative and depends on the Coulombic part $\text{Re}[\varphi_1]$ of the Faraday tensor. However, in the presence of electromagnetic fields at $\mathscr{I}$, the usual expression in equation (5.133) cannot be the full Wald-Zoupas charge of the theory, as it does not satisfy equation (5.122) with the full $\boldsymbol{\Theta}[\boldsymbol{\Phi}; \delta_\xi \boldsymbol{\Phi}]$ in equation (5.131), which includes the electromagnetic contribution $\boldsymbol{\Theta}_{\text{EM}}[\boldsymbol{A}; \delta_\xi \boldsymbol{A}]$.

[15] Note that for general supertranslations the "linkage" charges and fluxes do not equal the ones obtained from Hamiltonian methods [18], nor from the Wald-Zoupas prescription; see [22].

[16] In the Newman-Penrose notation, the Weyl tensor terms appearing in (5.133) are $\text{Re}[\Psi_2]$ and Ψ_1. Their derivatives on $\mathscr{I}$ along n^a are determined by the Bianchi identities given in equations (9.10.5) and (9.10.6) of [130].

Our goal now is to define the electromagnetic contribution $\mathcal{Q}_{\text{EM}}[\boldsymbol{\xi}; S]$ to the Wald-Zoupas charge such that $\mathcal{Q}_{\text{GR}}[\boldsymbol{X}; S] + \mathcal{Q}_{\text{EM}}[\boldsymbol{\xi}; S]$ satisfies equation (5.122) with the full $\boldsymbol{\Theta}[\boldsymbol{\Phi}; \delta_\xi \boldsymbol{\Phi}]$ given by equation (5.131). From equation (5.131), we have for $\boldsymbol{\Theta}_{\text{EM}}[\boldsymbol{A}; \delta_\xi \boldsymbol{A}]$ that

$$\int_{\Delta\mathscr{I}} \boldsymbol{\Theta}_{\text{EM}}[\boldsymbol{A}; \delta_\xi \boldsymbol{A}] = -\frac{1}{4\pi} \int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3 \; q^{ab} \mathcal{E}_a (\pounds_X A_b + \mathscr{D}_b \lambda). \tag{5.135}$$

This is precisely the flux $\mathcal{F}_{\text{N}}[\boldsymbol{\xi}; \Delta\mathscr{I}]$ of the Noether current of electromagnetism in equation (5.62). This relation arises due to our asymptotic conditions, which imply that $\boldsymbol{\Theta}_{\text{EM}}[\boldsymbol{A}; \delta \boldsymbol{A}] \mathrel{\hat{=}} \underleftarrow{\boldsymbol{\theta}}_{\text{EM}}[\boldsymbol{A}; \delta \boldsymbol{A}]$, where the right-hand side is the pullback of the symplectic potential of electromagnetism on a non-dynamical background given in equation (5.10). It also follows that $\underleftarrow{\boldsymbol{\eta}[\boldsymbol{\xi}]} \mathrel{\hat{=}} 0$ [see equation (5.21)], and thus $\boldsymbol{\Theta}_{\text{EM}}[\boldsymbol{A}; \delta_\xi A]$ is simply the pullback of the Noether current $\boldsymbol{J}_{\text{N}}[\boldsymbol{\xi}]$ for electromagnetism. The contribution of the electromagnetic field to the flux of the Wald-Zoupas charge is, in fact, not the stress-energy current, but the Noether current. This flux contribution is the same as the one obtained by Ashtekar and Streubel in equation (2.18) of [18]. However, there, the boundary term containing the Coulombic contribution $\text{Re}[\varphi_1]$ was dropped when converting to the stress-energy expression in their equation (2.19). This was valid in their context, as they considered only source-free solutions on Minkowski spacetime (so that $\text{Re}[\varphi_1]$ necessarily vanishes); for the more general scenario we are interested in, this boundary term is important and differentiates the Noether and stress-energy current.

From the previous computations, we can relate this electromagnetic contribution to the Wald-Zoupas flux to the stress-energy tensor using equations (5.68) and (5.69), obtaining

$$\mathcal{Q}_{\text{EM}}[\boldsymbol{\xi}; S_2] - \mathcal{Q}_{\text{EM}}[\boldsymbol{\xi}; S_1] = -\int_{\Delta\mathscr{I}} \left\{ \boldsymbol{\Theta}_{\text{EM}}[\boldsymbol{A}; \delta_\xi \boldsymbol{A}] - T_{ab} n^a X^b \boldsymbol{\epsilon}_3 \right\}, \tag{5.136}$$

where we have defined

$$\mathcal{Q}_{\text{EM}}[\boldsymbol{\xi}; S] \equiv \frac{1}{2\pi} \int_S \boldsymbol{\epsilon}_2 \, \text{Re}[\varphi_1](\lambda + X^a A_a), \tag{5.137}$$

which are essentially equation (5.69) and the integral of the electromagnetic Noether charge on the cross-section S in equation (5.25), respectively. Consequently, from equations (5.134) and (5.136), it follows that $\mathcal{Q}[\boldsymbol{\xi}; S] = \mathcal{Q}_{\text{GR}}[\boldsymbol{X}; S] + \mathcal{Q}_{\text{EM}}[\boldsymbol{\xi}; S]$ satisfies

$$\mathcal{F}[\boldsymbol{\xi}; \Delta\mathscr{I}] = -\int_{\Delta\mathscr{I}} \boldsymbol{\Theta}[\boldsymbol{\Phi}; \delta_\xi \boldsymbol{\Phi}] = \mathcal{Q}[\boldsymbol{\xi}; S_2] - \mathcal{Q}[\boldsymbol{\xi}; S_1]. \tag{5.138}$$

The electromagnetic contribution $\mathcal{Q}_{\rm EM}[\boldsymbol{\xi}; S] = 0$ when the Faraday tensor F_{ab} vanishes, and since $\mathcal{Q}_{\rm GR}[\boldsymbol{\xi}; S] = 0$ in Minkowski spacetime, the full Wald-Zoupas charge $\mathcal{Q}[\boldsymbol{\xi}; S]$ also vanishes in Minkowski spacetime.

In summary, the Wald-Zoupas charge for Einstein-Maxwell theory is

$$\mathcal{Q}[\boldsymbol{\xi}; S] = \mathcal{Q}_{\rm GR}[\boldsymbol{X}; S] + \mathcal{Q}_{\rm EM}[\boldsymbol{\xi}; S], \tag{5.139}$$

with the individual terms given by equations (5.133) and (5.137), respectively. The fluxes of the individual terms $\mathcal{Q}_{\rm GR}[\boldsymbol{X}; S]$ and $\mathcal{Q}_{\rm EM}[\boldsymbol{\xi}; S]$ depend on the stress-energy and *cannot* be determined purely from the radiative modes at null infinity. However, from equations (5.134) and (5.136), these contributions cancel exactly, and so the flux of the full Wald-Zoupas charge $\mathcal{Q}[\boldsymbol{\xi}; S]$ can be determined from the radiative modes alone.

As mentioned above, the flux $\mathcal{F}[\boldsymbol{\xi}; \mathscr{I}]$ is a Hamiltonian generator on the full radiative phase space of $\mathscr{I}$, corresponding to the symmetry $\boldsymbol{\xi}$. Along $\mathscr{I}$, as $u \to \pm\infty$, we have

$$N_{ab} = O(1/|u|^{1+\epsilon}), \qquad \mathcal{E}_a = O(1/|u|^{1+\epsilon}), \tag{5.140}$$

for some $\epsilon > 0$, while τ_{ab} and δA_a have finite limits as $u \to \pm\infty$. Note that these conditions are preserved by the asymptotic symmetries. Furthermore, they also ensure that the integral over all of $\mathscr{I}$ of the pullback of the symplectic current [equation (5.129)] is finite, and so we have a well-defined symplectic form on the radiative phase space on $\mathscr{I}$. Since X^a grows at most linearly in u, from equation (5.131) we have that

$$\lim_{u \to \pm\infty} \boldsymbol{X} \cdot \boldsymbol{\Theta}[\boldsymbol{\Phi}; \delta\boldsymbol{\Phi}] = 0, \tag{5.141}$$

and from equation (5.123),

$$\delta\mathcal{F}[\boldsymbol{\xi}; \mathscr{I}] = -\int_{\mathscr{I}} \underleftarrow{\boldsymbol{\omega}}[\boldsymbol{\Phi}; \delta\boldsymbol{\Phi}, \delta_\xi \boldsymbol{\Phi}], \tag{5.142}$$

for all perturbations $\delta\boldsymbol{\Phi}$ and all backgrounds $\boldsymbol{\Phi}$. Thus, the Wald-Zoupas flux acts as a Hamiltonian generator of the corresponding symmetry on the radiative phase space of Einstein-Maxwell theory on all of $\mathscr{I}$.[17]

There are several interesting consequences of this result. First, let us consider the behavior of the Wald-Zoupas charges under a gauge transformation $A_a \mapsto A_a + \nabla_a \Lambda$ with $\pounds_n \Lambda \mathrel{\hat{=}} 0$, so that

[17] If one instead defines the flux associated with a BMS symmetry by the right-hand side of (5.133), then such a flux is *not* a Hamiltonian generator in Einstein-Maxwell theory.

$n^a A_a \mathrel{\hat{=}} 0$ ((5.38)) is preserved. The gravitational contribution $\mathcal{Q}_{\rm GR}[\boldsymbol{X}; S]$ is, of course, unaffected by this transformation. Similarly, the electromagnetic contribution $\mathcal{Q}_{\rm EM}[\boldsymbol{\xi}; S]$ [in equation (5.137)] is invariant whenever the asymptotic symmetry $\boldsymbol{\xi}$ is either a pure gauge symmetry ($X^a = 0$) or a pure supertranslation ($X^a = f n^a$). However, the charge contribution $\mathcal{Q}_{\rm EM}[(\boldsymbol{Y}, 0); S]$ for a "pure Lorentz symmetry" transforms non-trivially:

$$\mathcal{Q}_{\rm EM}[(\boldsymbol{Y}, 0); S] \mapsto \mathcal{Q}_{\rm EM}[(\boldsymbol{Y}, 0); S] + \frac{1}{2\pi} \int_S \boldsymbol{\epsilon}_2 \ {\rm Re}[\varphi_1] \pounds_Y \Lambda. \tag{5.143}$$

The second term on the right-hand side is the charge $\mathcal{Q}_{\rm EM}[(0, \pounds_Y \Lambda); S]$ of a pure gauge symmetry $\pounds_Y \Lambda$. Thus, under gauge transformation, the electromagnetic contribution to the charge of a Lorentz symmetry shifts by the charge of a pure gauge symmetry. This is due to the fact that the action of a "pure Lorentz symmetry" $(\boldsymbol{Y}, 0)$ is not well-defined independently of the choice of gauge for A_a. This is similar to the transformation of the Lorentz charges under a supertranslation, and it essentially arises from the fact that the asymptotic symmetry algebra is a semidirect sum of the BMS algebra with the Lie ideal of gauge transformations. In the usual BMS algebra for vacuum GR, there is no unique Lorentz subalgebra but instead infinitely many Lorentz subalgebras which are related to each other by supertranslations. Similarly, in Einstein-Maxwell theory, there is no unique action of the Lorentz algebra on the vector potential A_a at $\mathscr{I}$, but instead infinitely many such actions of the Lorentz algebra which are all related by the asymptotic gauge symmetries. Note, however, that taking into account the change of the representation of $\boldsymbol{\xi}$ in terms of X^a and λ, the charge $\mathcal{Q}_{\rm EM}[\boldsymbol{\xi}; S]$ is invariant under gauge transformations as follows from equation (5.5). Essentially, under $A_a \mapsto A_a + \nabla_a \Lambda$, a "pure Lorentz symmetry" is not invariant but transforms as

$$(\boldsymbol{Y}, 0) \mapsto (\boldsymbol{Y}, -\pounds_Y \Lambda). \tag{5.144}$$

The transformation of the "pure Lorentz" charge in equation (5.143) is exactly compensated by the transformation of the "pure Lorentz" symmetry used to compute the charge.

The gravitational fields do not contribute to the Wald-Zoupas charge of a pure gauge symmetry $(0, \lambda)$, which is given by

$$\mathcal{Q}[(0, \lambda); \Delta\mathscr{I}] = \mathcal{Q}_{\rm EM}[(0, \lambda); S] \equiv \frac{1}{2\pi} \int_S \boldsymbol{\epsilon}_2 \,{\rm Re}[\varphi_1] \ \lambda, \tag{5.145}$$

with the flux

$$\mathcal{F}[(0,\lambda);\Delta\mathscr{I}] = \frac{1}{4\pi}\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3 \; q^{ab}\mathcal{E}_a \mathscr{D}_b \lambda. \tag{5.146}$$

For constant λ, the flux vanishes across any region $\Delta\mathscr{I}$, and the charge is proportional to the total conserved Coulomb charge. For a general λ (that is, λ is a function on $\mathbb{S}^2$) this charge is the "soft charge" of the electromagnetic fields (see [20, 16], for example).

Next, consider the charge associated with a supertranslation $\boldsymbol{\xi} = (f\boldsymbol{n}, 0)$. In this case, the electromagnetic contribution $\mathcal{Q}_{\rm EM}[\boldsymbol{\xi}; S]$ to the charge vanishes, since $n^a A_a \mathrel{\widehat{=}} 0$, and so the supermomentum charge is given by the same expression as in vacuum GR. Similarly, from (5.136) the electromagnetic contribution to the flux of supermomentum is also

$$-\int_{\Delta\mathscr{I}} \boldsymbol{\Theta}_{\rm EM}[\boldsymbol{A}; \delta_\xi \boldsymbol{A}] = -\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3 \; f T_{ab} n^a n^b = -\frac{1}{4\pi}\int_{\Delta\mathscr{I}} \boldsymbol{\epsilon}_3 \; f \mathcal{E}_a \mathcal{E}^a. \tag{5.147}$$

Thus, the electromagnetic fields do not contribute to the supermomentum charge and contribute to the supermomentum flux only through the asymptotic stress-energy tensor, which is purely radiative for supertranslations.

The situation is different for charges associated with a Lorentz symmetry $\boldsymbol{\xi} \mathrel{\widehat{=}} (\boldsymbol{Y}, 0)$. In this case, the electromagnetic fields contribute an additional term to the Wald-Zoupas charge given by

$$\mathcal{Q}_{\rm EM}[(\boldsymbol{Y}, 0); S] \equiv \frac{1}{2\pi}\int_S \boldsymbol{\epsilon}_2 \; \text{Re}[\varphi_1] Y^a A_a. \tag{5.148}$$

In section 5.3.2.2, we compute this term for a Kerr-Newman black hole and for a spinning charged sphere: in the case of the former, we find that this contribution is vanishing, although it does *not* vanish for the latter. A similar contribution to the angular momentum due to electromagnetic fields is also present at spatial infinity in stationary-axisymmetric spacetimes [158, 159, 133]. Thus, the electromagnetic contribution in equation (5.148) would also be relevant to show that the Lorentz charges defined on future null infinity coincide with those defined at spatial infinity and at past null infinity, as conjectured in [155].

5.3.2.2 Examples

In this section, we give two examples of the electromagnetic contribution to the Wald-Zoupas charge $\mathcal{Q}_{\rm EM}[(\boldsymbol{Y}, 0); S]$ of an asymptotic Lorentz symmetry Y^a. This contribution vanishes for the first example of Kerr-Newman spacetimes, while it is nonzero for the second example of a spinning charged sphere with variable angular velocity.

First, we consider the case of the Kerr-Newman spacetime. The line element of the (physical) Kerr-Newman metric in Boyer-Lindquist coordinates (t, r, θ, ϕ) is given by (see Appendix D.1 of [75])

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2 + \frac{r^2 + a^2 - \Delta}{\Sigma}(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi)^2 + (r^2 + a^2)\sin^2\theta \mathrm{d}\phi^2, \tag{5.149}$$

where

$$\Sigma \equiv r^2 + a^2\cos^2\theta, \qquad \Delta \equiv r^2 - 2Mr + a^2 + Q^2. \tag{5.150}$$

Note that this is the same as equation (2.1), with simply a different value of Δ. Since we wish consider the limit to $\mathscr{I}$, it is more convenient to introduce the *outgoing* null coordinates (u, r, θ, ϕ), with u defined by

$$\mathrm{d}u = \mathrm{d}t - \frac{r^2 + a^2}{\Delta}\mathrm{d}r. \tag{5.151}$$

The (physical) Kinnersley tetrad—normalized such that $\hat{l}^a \hat{n}_a = -1$ and $\hat{m}^a \hat{\bar{m}}_a = 1$—in these coordinates is

$$\hat{l}^a = (\partial_r)^a + \frac{a}{\Delta}(\partial_\phi)^a, \tag{5.152a}$$

$$\hat{n}^a = \frac{r^2 + a^2}{\Sigma}(\partial_u)^a - \frac{\Delta}{2\Sigma}(\partial_r)^a + \frac{a}{2\Sigma}(\partial_\phi)^a, \tag{5.152b}$$

$$\hat{m}^a = \frac{ia\sin\theta}{\sqrt{2}(r + ia\cos\theta)}(\partial_r)^a + \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left[(\partial_\theta)^a + \frac{i}{\sin\theta}(\partial_\phi)^a\right]. \tag{5.152c}$$

The vector potential in these null coordinates is

$$\hat{\boldsymbol{A}} = -\frac{rQ}{\Sigma}\left(\mathrm{d}u + \frac{r^2 + a^2}{\Delta}\mathrm{d}r - a\sin^2\theta \mathrm{d}\phi\right), \tag{5.153}$$

which satisfies the Lorenz gauge condition $\hat{\nabla}^a \hat{A}_a = 0$.

To take the limit to $\mathscr{I}$, we use the conformal factor $\Omega = r^{-1}$ and use Ω as a coordinate, instead of r. It can be verified that the unphysical metric $g_{ab} = \Omega^2 \hat{g}_{ab}$ is smooth in the limit to $\mathscr{I}$ (that is, as $\Omega \to 0$ with fixed u, θ, and ϕ). The unphysical tetrad $(l^a, n^a, m^a, \bar{m}^a)$, defined by

$$l^a \equiv \Omega^{-2}\hat{l}^a = (\partial_\Omega)^a + O(\Omega), \tag{5.154a}$$

$$n^a \equiv \hat{n}^a = (\partial_u)^a + O(\Omega), \tag{5.154b}$$

$$m^a \equiv \Omega^{-1}\hat{m}^a = \frac{1}{\sqrt{2}}\left[(\partial_\theta)^a + \frac{i}{\sin\theta}(\partial_\phi)^a\right] + O(\Omega), \tag{5.154c}$$

is also smooth at $\mathscr{I}$. The unphysical n^a defined in equation (5.154) coincides with the normal $n^a = g^{ab}\nabla_b\Omega$ at $\mathscr{I}$ to leading order, but not at $O(\Omega)$, as it does not satisfy the Bondi condition.

The vector potential $A_a = \hat{A}_a$ in equation (5.153) is not smooth at $\mathscr{I}$, since $l^a A_a$ diverges as $\Omega \to 0$. However, instead, consider the vector potential A'_a related to that in equation (5.153) by a gauge transformation:

$$A'_a = A_a - \nabla_a(Q \ln \Omega). \tag{5.155}$$

This new vector potential A'_a is no longer in Lorenz gauge (in the physical spacetime), but *is* smooth at $\mathscr{I}$; it also satisfies the outgoing radiation gauge condition $n^a A'_a \mathrel{\hat{=}} 0$. Henceforth, we use this smooth vector potential on $\mathscr{I}$ and drop the "prime" from the notation.

On $\mathscr{I}$, the Lorentz vector fields Y^a are spanned by the tetrads m^a and $\bar{m}^a$. A direct computation using equations (5.153), (5.154), and (5.155) gives $m^a A_a \mathrel{\hat{=}} 0$, and consequently $Y^a A_a \mathrel{\hat{=}} 0$ for all Lorentz vector fields. Thus, in the Kerr-Newman spacetime, the electromagnetic contribution to the Lorentz charges vanishes; that is, $\mathcal{Q}_{\text{EM}}[(\boldsymbol{Y}, 0); S] = 0$. In particular, the angular momentum of the Kerr-Newman black hole computed using the Wald-Zoupas charge [with $Y^a \equiv (\partial_\phi)^a$] gets no additional contribution from the electromagnetic fields, and is thus given by the standard value Ma (see, for example, [185]).[18]

The above computation of the Lorentz charges in Kerr-Newman spacetimes does not, of course, mean that the electromagnetic contribution to the Wald-Zoupas charge for angular momentum will always vanish. An explicit example for which this contribution is nonzero is considered in [39]: a thin spherical shell in Minkowski spacetime, with radius R and charge Q, spinning on a central axis with a time-dependent angular velocity $\omega(t)$. The time-dependent dipole moment of the spherical shell is given by $m(t) = \frac{1}{3}QR^2\omega(t)$. Furthermore, [39] also assumes that the characteristic timescale of variation of the magnetic dipole moment is much greater that the light-travel time $\tau = R$ across (half) the sphere, that is,

$$\frac{\mathrm{d}m}{\mathrm{d}t} \ll m(t)/\tau. \tag{5.156}$$

This is clearly not a solution to the source-free Maxwell equations, as there is a source current. However, as this source current is compact, our analysis still applies. We do not attempt to solve the

[18] To calculate the Wald-Zoupas charge using equation (5.133), one needs to be careful to use a tetrad with an n^a which satisfies the Bondi condition [equations (5.31) and (5.32)], and not the tetrad in equation (5.154).

full Einstein-Maxwell equations for this system. Thus, the electromagnetic field in this section should be thought of as a perturbation generated by the charged sphere on the background Minkowski spacetime.

The relevant null tetrads at $\mathscr{I}$ in Minkowski spacetime can be constructed in the same manner as in the Kerr-Newman spacetime by taking $M = a = Q = 0$. To get a smooth vector potential at $\mathscr{I}$, one again needs to perform a gauge transformation as in equation (5.155) which takes us out of the Lorenz gauge used in [39]. Then, from the explicit computations in [39], it can be shown that

$$\mathrm{Re}[\varphi_1] \hat{=} \frac{1}{2}Q, \qquad m^a A_a \hat{=} \frac{i}{\sqrt{2}}\Gamma^{(0)}(u)\sin\theta, \tag{5.157}$$

where $u = t - r$ is the retarded time coordinate and we have taken the rotation axis for the sphere to be along the z-axis. With the assumption given in equation (5.156), the function $\Gamma^{(0)}(u)$ is given by

$$\Gamma^{(0)}(u) \equiv \frac{\mathrm{d}m}{\mathrm{d}u} + \frac{1}{10}\tau^2\frac{\mathrm{d}^3 m}{\mathrm{d}u^3} + \frac{1}{280}\tau^4\frac{\mathrm{d}^5 m}{\mathrm{d}u^5} + \cdots, \tag{5.158}$$

where $\cdots$ denotes higher-order terms.

Now, the rotational Killing vector field along the z-axis is given by

$$(\partial_\phi)^a = -\frac{i}{\sqrt{2}}\sin\theta(m^a - \bar{m}^a). \tag{5.159}$$

Thus, using equations (5.157) and (5.158), we can compute the electromagnetic contribution in equation (5.137) to the charge of ∂_ϕ—that is, the angular momentum in the z direction—on a cross-section S_u of constant u:

$$\mathcal{Q}_{\mathrm{EM}}[(\partial_\phi, 0); S_u] = \frac{2}{3}Q\Gamma^{(0)}(u). \tag{5.160}$$

Thus, we expect that generic non-stationary electromagnetic fields will contribute a non-vanishing $\mathcal{Q}_{\mathrm{EM}}$ to the Wald-Zoupas charge for asymptotic Lorentz symmetries.

5.4 | Discussion

In this chapter, we analyzed the fluxes of electromagnetic fields associated with the asymptotic symmetries at null infinity in any asymptotically flat spacetime. We first considered electromagnetism on a non-dynamical background, defining three different currents which are naturally associated with vector fields on the background spacetime. When the vector field is a Killing vector field of the background spacetime, each of these currents is conserved, and they differ only by boundary terms. A similar situation occurs at null infinity, under the weaker assumption that the vector field is an asymptotic symmetry element of the BMS algebra. In this case, each of the three currents can be used to construct fluxes associated with the asymptotic symmetry algebra through a given region of null infinity. While two of the currents, the Noether and canonical current fluxes, are completely determined by the radiative degrees of freedom of the electromagnetic fields, the flux associated with the asymptotic Lorentz symmetries defined by the stress-energy current also depends on the Coulombic part of the electromagnetic field. Thus, if the stress-energy flux for a rotational symmetry is interpreted as the flux of angular momentum through null infinity, then it cannot be determined from the radiative degrees of freedom alone [19, 20]. Furthermore, none of these fluxes can be considered as the difference of charges evaluated on cross-sections of null infinity, as on a non-dynamical background spacetime, there is, in general, no notion of an energy or angular momentum of the electromagnetic fields at a particular "time" defined by a cross-section of null infinity. Therefore, there is no obvious way to decide which of these currents defines the flux of energy or angular momentum.

To clarify this, we coupled electromagnetism to general relativity and considered the full Einstein-Maxwell theory at null infinity. This theory is diffeomorphism-invariant, and so there exist charges whose differences are given by fluxes. Specifically, the general prescription of Wald and Zoupas [178] defines, for a given asymptotic symmetry, both the charge on a cross-section of $\mathscr{I}$ and the flux, which represents the change in this charge. If one assumes the charge expression for vacuum GR to be the definition of the charge in Einstein-Maxwell theory as well [see equation (5.133)], then the additional term that the electromagnetic fields contribute to its flux is the stress-energy flux [equation (5.134)]. As in the case with a non-dynamical metric, this contribution depends on the Coulombic part of the Maxwell field for asymptotic Lorentz symmetries. However, the full Wald-Zoupas charge for Einstein-Maxwell theory contains an additional contribution to the charge due to the electromagnetic fields [equation (5.137)]. This additional contribution vanishes for asymptotic supertranslations. It also vanishes for Lorentz symmetries in the Kerr-Newman spacetime. In general, however, for non-stationary electromagnetic fields, this additional contribution is nonzero. The flux of the full Wald-Zoupas charge in Einstein-Maxwell theory with this additional contribution from the electromagnetic fields is determined by the radiative fields alone. The full Wald-Zoupas charge naturally absorbs the Coulombic information contained in the stress-energy flux, and so the contribution of the electromagnetic fields to the Wald-Zoupas flux is determined by the Noether current flux and depends only on the radiative fields on $\mathscr{I}$.

In addition, we showed that, using the standard falloff conditions for the electromagnetic and gravitational fields near i^0 and i^+, the Wald-Zoupas flux also defines a Hamiltonian generator associated with the asymptotic symmetries on all of null infinity.

A similar analysis can also be carried out for other matter fields. For GR minimally coupled to a massless Klein-Gordon field or a conformally-coupled scalar field, the essential points have already been discussed by Wald and Zoupas in section VI of [178]. For such fields, the Wald-Zoupas charge is given by the same expression as in vacuum GR [equation (5.133)], and the scalar fields contribute to the flux only through the stress-energy tensor. However, for Einstein-Yang-Mills theory, we expect that there is an additional contribution to the Wald-Zoupas charge similar to the case of electromagnetic fields considered here. For general theories, it should *not* be expected that the matter contribution to the charge is the Noether charge or that the contribution to the flux is the Noether current. For instance, this expectation is already false in vacuum GR, where the Wald-Zoupas charge is, in general, *not* given by the Noether charge (that is, the Komar formula); see the discussion in [78, 22].

As noted before, a similar additional contribution to the angular momentum due to electromagnetic fields is also present at spatial infinity in stationary, axisymmetric spacetimes [158, 159, 133]. Thus, we expect that the electromagnetic contribution in equation (5.148) would also be relevant to show that the Lorentz charges defined on future null infinity coincide with those defined at spatial infinity and at past null infinity, as conjectured in [155].

Since the Wald-Zoupas flux is purely radiative and also the Hamiltonian generator on the radiative phase space of Einstein-Maxwell theory, it can be quantized using the asymptotic quantization methods in [16].

The Wald-Zoupas prescription can also be applied to finite null surfaces in vacuum GR [50]. For Einstein-Maxwell theory at finite null surfaces, we expect that there is a similar contribution to the charges and fluxes associated with finite null boundary symmetries considered in [50] that arises from the electromagnetic fields. Such an analysis could also be useful in deriving conservation laws in Einstein-Maxwell theory through local regions bounded by a causal diamond similar to those in vacuum GR [51].

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